LECTURE 5 · SOFT COMPUTING

From Fuzzy Relations to Fuzzy Control

Relations, propositions, implications, inference, defuzzification and fuzzy logic controllers



$\mu R(x, y)$

What you will be able to do

- **R** — Build fuzzy relations and compose them with max–min and max–product
- **P** — Evaluate the truth of fuzzy propositions and their connectives
- **→** — Turn an IF–THEN rule into a relation matrix (Mamdani, Larsen, Zadeh)
- **⊢** — Draw conclusions with generalized modus ponens and modus tollens
- **x*** — Defuzzify with λ-cut, maxima, centroid and weighted-average methods
- **C** — Design Mamdani and Takagi–Sugeno fuzzy logic controllers

Today’s path: from a relation to a control action

R

Relations

Grade how strongly x and y are linked, then chain links.

P

Propositions

“v is F” is true to the degree μF(v).

→

Implications

Store an IF–THEN rule as a relation matrix.

⊢

Inference

Derive new facts from rules: B′ = A′ ∘ R.

x*

Defuzzification

Turn a fuzzy result into one crisp number.

FLC

Controllers

Fuzzify, infer and defuzzify in every control cycle.

Built on lecture 4: every step today reuses min, max, the complement and the Cartesian product

$$\mu_{A\times B}(x,y) = \min(\mu_A(x),\ \mu_B(y))$$

01

Fuzzy relations

Crisp relations · fuzzy relations · operations · max–min and max–product composition

Example 1: A crisp relation from a rule

Given

$A = \{1, 2, 3, 4\}$ and $B = \{3, 5, 7\}$.

R is the relation "$b = a + 1$" on $A \times B$.

Required

(a) How many ordered pairs does $A \times B$ contain?

(b) List the pairs of R.

(c) Write R as a 0/1 relation matrix (rows a, columns b).

A crisp relation is a subset of A × B

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}$$

$$A \times B \neq B \times A, \qquad |A \times B| = |A| \cdot |B|$$

Example

$$A = \{1,2,3,4\}, \qquad B = \{3,5,7\}$$

$$R = \{(a, b) \in A \times B \mid b = a + 1\}$$

$$R = \{(2,3),\ (4,5)\}$$

A relation is the set of ordered pairs that satisfy a rule: a mapping from elements of A to elements of B.

a \ b	3	5	7
1	0	0	0
2	1	0	0
3	0	0	0
4	0	1	0

Relation matrix: 1 means the pair is in R, 0 means it is not.

A fuzzy relation grades each pair

$$R = \{\, ((x, y),\ \mu_R(x, y)) \mid (x, y) \in X \times Y \,\}$$

$$\mu_R : X \times Y \to [0,1]$$

- Crisp relation: every entry of the matrix is 0 or 1.
- Fuzzy relation: an entry can be any value in [0, 1], the strength of the link.
- Read a row: typhoid shows high temperature to degree 0.9 but a running nose only to 0.1.

disease	running nose	high temp.	shivering
typhoid	0.1	0.9	0.8
viral	0.2	0.9	0.7
cold	0.9	0.4	0.6

$R \subseteq$ diseases × symptoms: how strongly each disease shows each symptom

Only the entries change. Every crisp operation carries over to fuzzy relations.

Example 2: Building fuzzy relations

Given

$A = \{(a_1, 0.2), (a_2, 0.7), (a_3, 0.4)\}$ and $B = \{(b_1, 0.5), (b_2, 0.6)\}$.

$X = \{3, 4, 5\}$, $Y = \{3, 4, 5, 6, 7\}$ and R = "y is much greater than x", with $\mu_R(x, y) = (y - x) / 4$ for $y > x$ and 0 otherwise.

Required

(a) Build the fuzzy relation $A \times B$ using min.

(b) Build the 3×5 relation matrix of R.

Two ways to build a fuzzy relation

From two fuzzy sets

$$\mu_{A\times B}(x,y) = \min(\mu_A(x),\ \mu_B(y))$$

$$A = \{(a_1, 0.2),\ (a_2, 0.7),\ (a_3, 0.4)\}$$

$$B = \{(b_1, 0.5),\ (b_2, 0.6)\}$$

A × B	b_1	b_2
a_1	0.2	0.2
a_2	0.5	0.6
a_3	0.4	0.4

From a formula on X × Y

$$\mu_R(x,y) = \begin{cases} \dfrac{y-x}{4} & y > x \\ 0 & y \le x \end{cases}$$

R = “y is much greater than x”, X = {3, 4, 5}, Y = {3, …, 7}

x \ y	3	4	5	6	7
3	0	0.25	0.5	0.75	1
4	0	0	0.25	0.5	0.75
5	0	0	0	0.25	0.5

Example 3: Operations on fuzzy relations

Given

R and P are fuzzy relations on $X \times Y$, with $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$.

Required

(a) Find $R \cup P$.

(b) Find $R \cap P$.

(c) Find R^c.

(d) Can R be joined by union with a relation S on $Y \times Z$? Why or why not?

R

	y_1	y_2
x_1	0.5	0.1
x_2	0.2	0.9
x_3	0.8	0.6

P

	y_1	y_2
x_1	0.1	0.2
x_2	0.2	0.5
x_3	0.3	0.4

Union, intersection and complement work cell by cell

∪ Union

$$\mu_{R \cup P} = \max(\mu_R, \mu_P)$$

∩ Intersection

$$\mu_{R \cap P} = \min(\mu_R, \mu_P)$$

¬ Complement

$$\mu_{R^c} = 1 - \mu_R$$

R

	y_1	y_2
x_1	0.5	0.1
x_2	0.2	0.9
x_3	0.8	0.6

P

	y_1	y_2
x_1	0.1	0.2
x_2	0.2	0.5
x_3	0.3	0.4

R ∪ P

	y_1	y_2
x_1	0.5	0.2
x_2	0.2	0.9
x_3	0.8	0.6

R ∩ P

	y_1	y_2
x_1	0.1	0.1
x_2	0.2	0.5
x_3	0.3	0.4

R^c

	y_1	y_2
x_1	0.5	0.9
x_2	0.8	0.1
x_3	0.2	0.4

Union and intersection need both relations on the same X × Y. The complement is unary: it needs one relation.

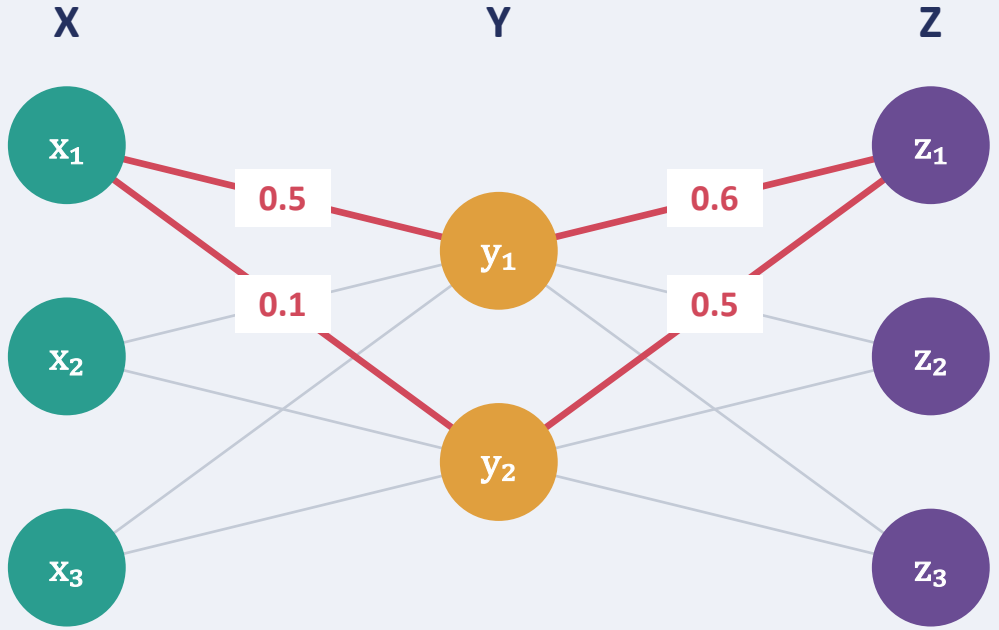
Max–min composition chains two relations

$$\mu_{R\circ S}(x,z) = \max_{y\in Y} \min(\mu_R(x,y),\ \mu_S(y,z))$$

$$R \subseteq X\times Y, \quad S \subseteq Y\times Z, \quad R\circ S \subseteq X\times Z$$

- For entry (x, z), pair row x of R with column z of S.
- Take the min of each pair, then the max of those mins.
- It is a matrix product with × replaced by min and + by max.
- Y must be shared, like the inner dimensions of a matrix product.

Composition is the engine of fuzzy inference (section 04).



$$\min(0.5,\ 0.6) = 0.5, \qquad \min(0.1,\ 0.5) = 0.1$$
$$\mu_{R\circ S}(x_1, z_1) = \max(0.5,\ 0.1) = 0.5$$

Example 4: Max–min composition

Given

R is a fuzzy relation on $X \times Y$ and S a fuzzy relation on $Y \times Z$.

$X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$, $Z = \{z_1, z_2, z_3\}$.

Required

(a) Find $T = R \circ S$ by max–min composition.

(b) Show the full working for the cells (x_1, z_1) and (x_3, z_2).

R

	y_1	y_2
x_1	0.5	0.1
x_2	0.2	0.9
x_3	0.8	0.6

S

	z_1	z_2	z_3
y_1	0.6	0.4	0.7
y_2	0.5	0.8	0.9

Composition, worked cell by cell

R on X × Y

R	y_1	y_2
x_1	0.5	0.1
x_2	0.2	0.9
x_3	0.8	0.6

$\circ$

S on Y × Z

S	z_1	z_2	z_3
y_1	0.6	0.4	0.7
y_2	0.5	0.8	0.9

$=$

T = R ∘ S on X × Z

T	z_1	z_2	z_3
x_1	0.5	0.4	0.5
x_2	0.5	0.8	0.9
x_3	0.6	0.6	0.7

$$\mu_T(x_1, z_1) = \max\{\min(0.5,0.6),\ \min(0.1,0.5)\} = \max\{0.5,\ 0.1\} = 0.5$$

$$\mu_T(x_3, z_2) = \max\{\min(0.8,0.4),\ \min(0.6,0.8)\} = \max\{0.4,\ 0.6\} = 0.6$$

Your turn: check (x_2, z_3) = max{min(0.2, 0.7), min(0.9, 0.9)} = 0.9.

Example 5: Max–min versus max–product

Given

R_1 is a fuzzy relation on $X \times Y$ and R_2 a fuzzy relation on $Y \times Z$ (Illustrated Handbook, Example 5).

Required

(a) Find $R_1 \circ R_2$ by max–min composition.

(b) Find $R_1 \circ R_2$ by max–product composition.

(c) Which cells differ? Why can max–product never exceed max–min?

R_1

	y_1	y_2
x_1	0.6	0.5
x_2	1	0.1
x_3	0	0.7

R_2

	z_1	z_2	z_3
y_1	0.7	0.3	0.4
y_2	0.9	0.1	0.6

Max–product: a softer composition

$$\mu_{R\circ S}(x,z) = \max_{y} \left(\mu_R(x,y) \cdot \mu_S(y,z)\right)$$

Inputs (Illustrated Handbook, Example 5)

R_1	y_1	y_2
x_1	0.6	0.5
x_2	1	0.1
x_3	0	0.7

R_2	z_1	z_2	z_3
y_1	0.7	0.3	0.4
y_2	0.9	0.1	0.6

- Product ≤ min, so max–product never exceeds max–min.
- For crisp 0/1 relations the two always agree.

Max–min $R_1 \circ R_2$

	z_1	z_2	z_3
x_1	0.6	0.3	0.5
x_2	0.7	0.3	0.4
x_3	0.7	0.1	0.6

Max–product (differences in red)

	z_1	z_2	z_3
x_1	**0.45**	**0.18**	**0.3**
x_2	0.7	0.3	0.4
x_3	**0.63**	**0.07**	**0.42**

Example 6: Plant → disease → symptom

Given

R grades how strongly each plant variety P_1–P_3 suffers each disease D_1–D_4.

S grades how strongly each disease shows each symptom S_1, S_2.

Required

(a) Find $R \circ S$ by max–min composition.

(b) How strongly does plant P_2 show symptom S_1, and through which disease does the strongest path run?

R: plant → disease

	D_1	D_2	D_3	D_4
P_1	0.1	0.3	0.5	0.7
P_2	0.4	0.2	0.8	0.9
P_3	0.6	0.8	0.3	0.2

S: disease → symptom

	S_1	S_2
D_1	0.9	0.1
D_2	0.2	0.3
D_3	0.5	0.6
D_4	0.7	0.2

Chaining relations: plant → disease → symptom

R: plant → disease

R	D_1	D_2	D_3	D_4
P_1	0.1	0.3	0.5	0.7
P_2	0.4	0.2	0.8	**0.9**
P_3	0.6	0.8	0.3	0.2

$\circ$

S: disease → symptom

S	S_1	S_2
D_1	0.9	0.1
D_2	0.2	0.3
D_3	0.5	0.6
D_4	**0.7**	0.2

$=$

R ∘ S: plant → symptom

T	S_1	S_2
P_1	0.7	0.5
P_2	**0.7**	0.6
P_3	0.6	0.3

$$\mu_{R\circ S}(P_2, S_1) = \max\{0.4 \wedge 0.9,\ 0.2 \wedge 0.2,\ 0.8 \wedge 0.5,\ 0.9 \wedge 0.7\} = \max\{0.4,\ 0.2,\ 0.5,\ 0.7\} = 0.7$$

Plant P_2 shows symptom S_1 with strength 0.7. The strongest path runs through disease D_4 (red cells).

02

Fuzzy propositions

Multi-valued logic · connectives · canonical form P : v is F

Two-valued logic grows into multi-valued logic

Crisp (two-valued) proposition

T(P) ∈ {0, 1}. “The sun rises in the east” is simply true: T(P) = 1.

Fuzzy (multi-valued) proposition

T(P) ∈ [0, 1]. “The mango is sweet” can be true to degree 0.7.

Three-valued logic: add ½

AND = min, OR = max, NOT a = 1 − a, a ⇒ b = max(1 − a, b), a = b gives 1 − |a − b|.

a	b	a ∧ b	a ∨ b	¬a	a ⇒ b	a = b
0	0	0	0	1	1	1
0	½	0	½	1	1	½
0	1	0	1	1	1	0
½	0	0	½	½	½	½
½	½	½	½	½	½	1
½	1	½	1	½	1	½
1	0	0	1	0	0	0
1	½	½	1	0	½	½
1	1	1	1	0	1	1

Example 7: Truth of compound propositions

Given

P: “Mary is efficient”, with T(P) = 0.8.

Q: “Ram is efficient”, with T(Q) = 0.6.

Required

(a) T(¬P): Mary is not efficient.

(b) T(P ∧ Q): Mary and Ram are efficient.

(c) T(P ∨ Q): Mary or Ram is efficient.

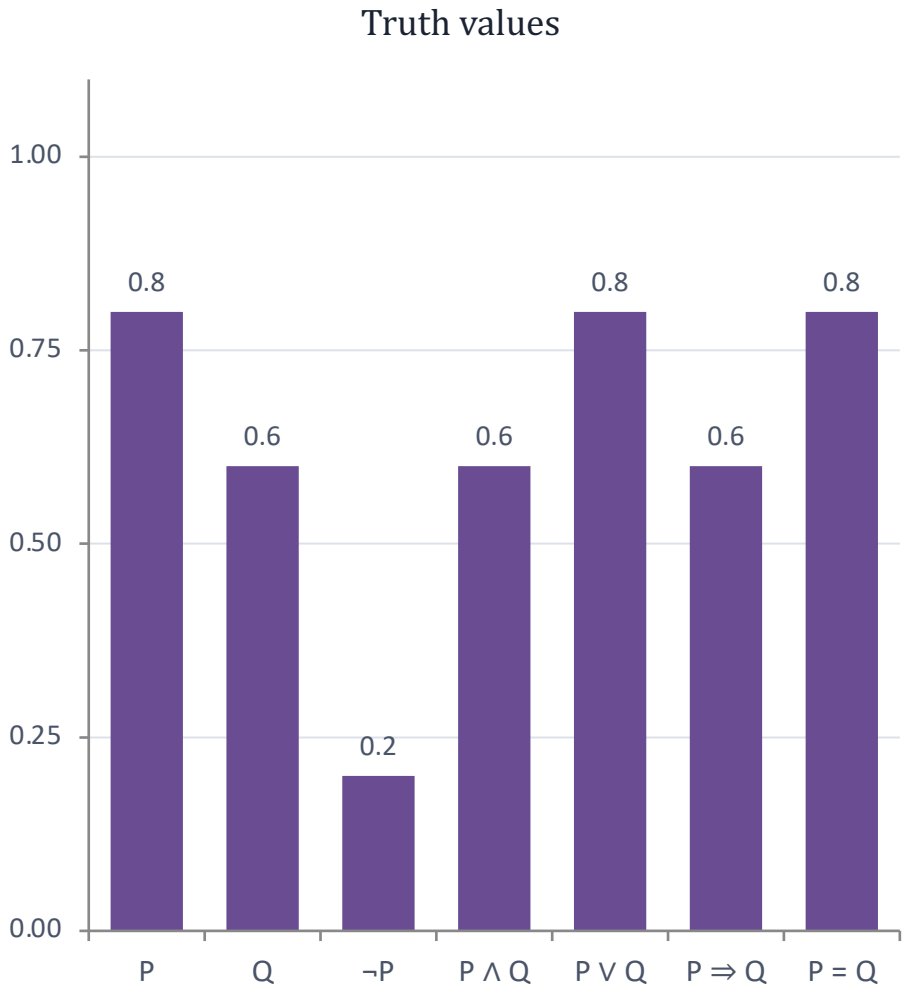
(d) T(P ⇒ Q): if Mary is efficient then so is Ram.

(e) T(P = Q).

Fuzzy connectives and a worked example

Symbol	Connective	Usage	Truth value
¬	NOT	$\neg P$	$1 - T(P)$
∨	OR	$P \vee Q$	$\max\{T(P), T(Q)\}$
∧	AND	$P \wedge Q$	$\min\{T(P), T(Q)\}$
⇒	IMPLICATION	$P \Rightarrow Q \ \ (\neg P \vee Q)$	$\max\{1 - T(P), T(Q)\}$
=	EQUALITY	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$	$1 - \lvert T(P) - T(Q) \rvert$

P: Mary is efficient, T(P) = 0.8. Q: Ram is efficient, T(Q) = 0.6.
"If Mary is efficient then so is Ram": T(P ⇒ Q) = max(1 − 0.8, 0.6) = 0.6.



Example 8: Truth of a canonical proposition

Given

A universe of five persons $X = \{x_1, x_2, x_3, x_4, x_5\}$.

intelligent = $\{(x_1, 0.3), (x_2, 0.4), (x_3, 0.1), (x_4, 0.6), (x_5, 0.9)\}$.

Required

(a) Write "x_4 is intelligent" in the canonical form P : v is F.

(b) Find its truth value T(P).

Canonical form: P : v is F

$$P\text{: } v \text{ is } F \qquad\qquad T(P) = \mu_F(v)$$

- v: a variable that takes values in a universe V.
- F: a fuzzy set on V, usually a linguistic term (hot, tall, intelligent).
- The truth of “v is F” is the grade of v in F.
- “20 °C is hot” is true to degree μhot(20).

$$T(x_4 \text{ is intelligent}) = \mu_{\text{intelligent}}(x_4) = 0.6$$



Propositions combine into rules

1 Proposition

x is A

A statement whose truth is μA(x).

2 Rule

IF x is A THEN z is C

Links two propositions.

3 Relation

$R_1 = A \times C \subseteq X \times Z$

Stored as a matrix over X × Z.

Compound antecedents ($R_1 = A \times C$, $R_2 = B \times C$)

IF x is A OR y is B THEN z is C: $R = R_1 \cup R_2$

IF x is A AND y is B THEN z is C: $R = R_1 \cap R_2$

03

Fuzzy implications

IF–THEN rules · rule matrices · Mamdani, Larsen and Zadeh interpretations

A fuzzy rule is an IF–THEN implication

IF x is A

antecedent (premise)

THEN y is B

consequent (conclusion)

$$R\colon\ A \to B, \qquad R \subseteq X \times Y$$

A binary fuzzy relation from A to B. Also called a fuzzy rule, an IF–THEN rule or a fuzzy conditional statement.

1

If pressure is high then temperature is low.

2

If mango is yellow then mango is sweet, else mango is sour.

3

If road is good then driving is smooth, else traffic is high.

Example 9: High temperature → low pressure

Given

A heating system follows the rule: IF temperature is high THEN pressure is low.

T = {20, 25, 30, 35, 40, 45, 50} and P = {1, 2, 3, 4}.

T_{HIGH} = {(20, 0.2), (25, 0.4), (30, 0.6), (35, 0.6), (40, 0.7), (45, 0.8), (50, 0.8)}

P_{LOW} = {(1, 0.8), (2, 0.8), (3, 0.6), (4, 0.4)}

Required

(a) Find the fuzzy relation R: $T_{HIGH} \rightarrow P_{LOW}$ using the Mamdani (min) implication, $\mu_R(t, p) = \min(\mu_T(t), \mu_P(p))$.

(b) What does the rule conclude about the pressure when the temperature is T = 40?

A rule as a matrix: high temperature → low pressure

High temperature T_{HIGH}

T	20	25	30	35	40	45	50
μ	0.2	0.4	0.6	0.6	0.7	0.8	0.8

Low pressure P_{LOW}

P	1	2	3	4
μ	0.8	0.8	0.6	0.4

Reading the rule at T = 40

$$P_{LOW}\,|_{T=40} = \{(1,0.7),\ (2,0.7),\ (3,0.6),\ (4,0.4)\}$$

Row 40 of R is the rule's answer about pressure when the temperature is 40.

T \ P	1	2	3	4
20	0.2	0.2	0.2	0.2
25	0.4	0.4	0.4	0.4
30	0.6	0.6	0.6	0.4
35	0.6	0.6	0.6	0.4
40	**0.7**	**0.7**	**0.6**	**0.4**
45	0.8	0.8	0.6	0.4
50	0.8	0.8	0.6	0.4

$R = T_{HIGH} \times P_{LOW}$ with min (the Mamdani form)

Worked example: building R one cell at a time

$\mu_R(t, p) = \min(\mu_T(t), \mu_P(p))$ *each cell is the smaller of its row grade and its column grade*

T (μT) \ P (μP)	P = 1 (0.8)	P = 2 (0.8)	P = 3 (0.6)	P = 4 (0.4)
T = 20 (0.2)	min(0.2, 0.8) = 0.2	min(0.2, 0.8) = 0.2	min(0.2, 0.6) = 0.2	min(0.2, 0.4) = 0.2
T = 25 (0.4)	min(0.4, 0.8) = 0.4	min(0.4, 0.8) = 0.4	min(0.4, 0.6) = 0.4	min(0.4, 0.4) = 0.4
T = 30 (0.6)	min(0.6, 0.8) = 0.6	min(0.6, 0.8) = 0.6	min(0.6, 0.6) = 0.6	min(0.6, 0.4) = 0.4
T = 35 (0.6)	min(0.6, 0.8) = 0.6	min(0.6, 0.8) = 0.6	min(0.6, 0.6) = 0.6	min(0.6, 0.4) = 0.4
T = 40 (0.7)	**min(0.7, 0.8) = 0.7**	**min(0.7, 0.8) = 0.7**	**min(0.7, 0.6) = 0.6**	**min(0.7, 0.4) = 0.4**
T = 45 (0.8)	min(0.8, 0.8) = 0.8	min(0.8, 0.8) = 0.8	min(0.8, 0.6) = 0.6	min(0.8, 0.4) = 0.4
T = 50 (0.8)	min(0.8, 0.8) = 0.8	min(0.8, 0.8) = 0.8	min(0.8, 0.6) = 0.6	min(0.8, 0.4) = 0.4

Two families of implication operators

$$\mu_R(x,y) = f(\mu_A(x),\ \mu_B(y)) = f(a,b)$$

A coupled with B

conjunction: R = A × B with a T-norm

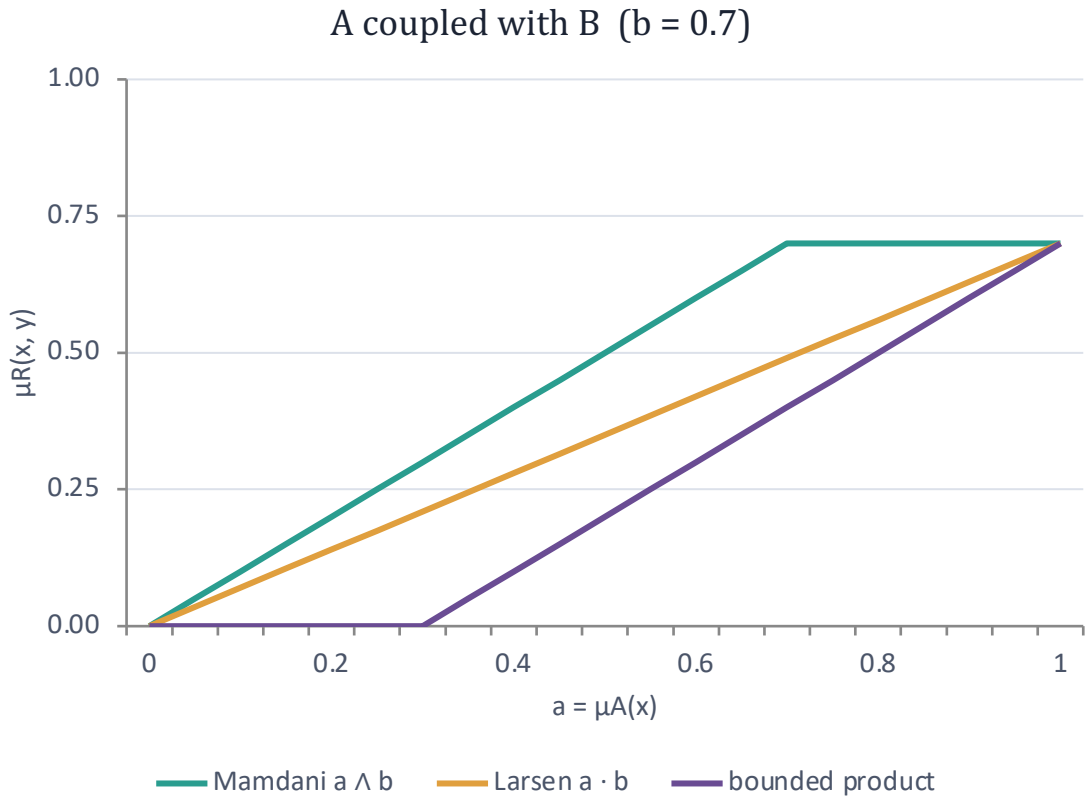
Mamdani (min)	$a \wedge b$
Larsen (product)	$a \cdot b$
Bounded product	$0 \vee (a + b - 1)$
Drastic product	$\begin{cases} a & b = 1 \\ b & a = 1 \\ 0 & \text{otherwise} \end{cases}$

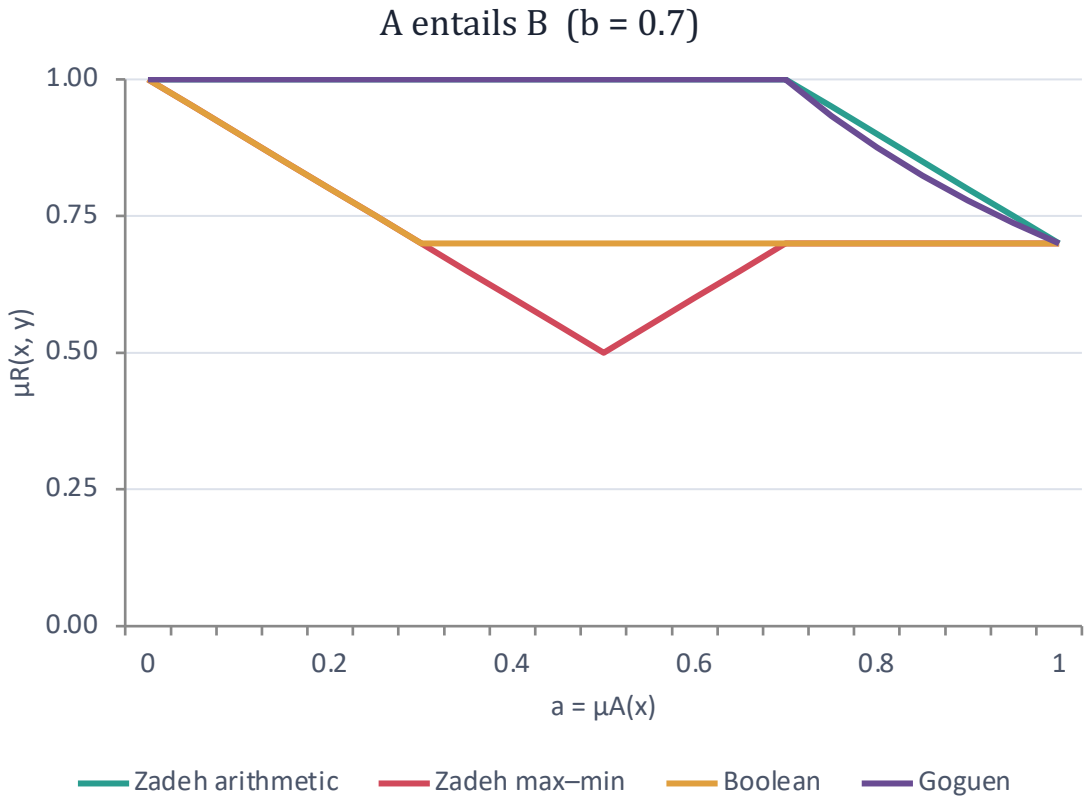
A entails B

implication: built with the complement

Zadeh arithmetic	$1 \wedge (1 - a + b)$
Zadeh max–min	$(1 - a) \vee (a \wedge b)$
Boolean	$(1 - a) \vee b$
Goguen	$\begin{cases} 1 & a \le b \\ b/a & a > b \end{cases}$

Same inputs, different relations





At a = 0 (antecedent false) the coupled operators give 0 and the entailment operators give 1: a false premise makes an implication vacuously true.

Example 10: Zadeh’s max–min rule

Given

$X = Y = \{a, b, c, d\}$ and the rule IF x is A THEN y is B.

$A = \{(a, 0), (b, 0.8), (c, 0.6), (d, 1)\}$

$B = \{(a, 0.2), (b, 1), (c, 0.8), (d, 0)\}$

Required

(a) Find $A \times B$ and $A^c \times Y$.

(b) Find the rule matrix $R = (A \times B) \cup (A^c \times Y)$.

(c) Explain why row a of R is all 1s and row d copies B.

Zadeh's max–min rule, worked

$$R_{mm} = (A \times B) \cup (\bar{A} \times Y), \qquad \mu_Y(y) = 1 \;\; \forall y \in Y$$

$$A = \{(a, 0),\ (b, 0.8),\ (c, 0.6),\ (d, 1)\}, \qquad B = \{(1,0.2),\ (2,1),\ (3,0.8),\ (4,0)\}$$

A × B

	1	2	3	4
a	0	0	0	0
b	0.2	0.8	0.8	0
c	0.2	0.6	0.6	0
d	0.2	1	0.8	0

∪

Ā × Y

	1	2	3	4
a	1	1	1	1
b	0.2	0.2	0.2	0.2
c	0.4	0.4	0.4	0.4
d	0	0	0	0

=

R (max–min rule)

	1	2	3	4
a	1	1	1	1
b	0.2	0.8	0.8	0.2
c	0.4	0.6	0.6	0.4
d	0.2	1	0.8	0

Row a: μA(a) = 0, so the rule makes no claim there and the row is all 1s. Row d: μA(d) = 1, so the row copies B.

Example 11: IF … THEN … ELSE

Given

X = Y = {a, b, c, d} and the rule IF x is A THEN y is B ELSE y is C.

A = {(a, 0), (b, 0.8), (c, 0.6), (d, 1)}

B = {(a, 0.2), (b, 1), (c, 0.8), (d, 0)}

C = {(a, 0), (b, 0.4), (c, 1), (d, 0.8)}

Required

(a) Find $A \times B$ and $A^c \times C$.

(b) Find the rule matrix $R = (A \times B) \cup (A^c \times C)$.

(c) Which rows of R follow B, which follow C, and which blend both?

IF … THEN … ELSE as one relation

$$R = (A \times B) \cup (\bar{A} \times C)$$

$$\mu_R(x, y) = \max[\min(\mu_A, \mu_B),\ \min(1 - \mu_A, \mu_C)]$$

$$C = \{(1,0),\ (2,0.4),\ (3,1),\ (4,0.8)\}$$

A × B

	1	2	3	4
a	0	0	0	0
b	0.2	0.8	0.8	0
c	0.2	0.6	0.6	0
d	0.2	1	0.8	0

Ā × C

	1	2	3	4
a	0	0.4	1	0.8
b	0	0.2	0.2	0.2
c	0	0.4	0.4	0.4
d	0	0	0	0

A and B as on the previous slide

R for IF x is A THEN y is B ELSE y is C

	1	2	3	4
a	0	0.4	1	0.8
b	0.2	0.8	0.8	0.2
c	0.2	0.6	0.6	0.4
d	0.2	1	0.8	0

Where A holds (row d) R follows B. Where A fails (row a) R follows C. Rows b and c blend both.

04

Fuzzy inference

Modus ponens and tollens · GMP and GMT · inference by composition

Classical inference rules

Modus ponens

$$\frac{P \qquad P \Rightarrow Q}{Q}$$

Given P and P ⇒ Q, conclude Q.

Modus tollens

$$\frac{P \Rightarrow Q \qquad \neg Q}{\neg P}$$

Given P ⇒ Q and not Q, conclude not P.

Chain rule

$$\frac{P \Rightarrow Q \qquad Q \Rightarrow R}{P \Rightarrow R}$$

Implications chain together.

Why modus ponens works in two-valued logic

$$\begin{aligned} P \wedge (P \Rightarrow Q) &= P \wedge (\neg P \vee Q) \\ &= (P \wedge \neg P) \vee (P \wedge Q) \\ &= 0 \vee (P \wedge Q) = P \wedge Q \;\Rightarrow\; Q \end{aligned}$$

Fuzzy logic keeps these patterns but lets premises and conclusions hold to a degree: the generalized forms on the next slide.

Generalized modus ponens and modus tollens

⊢ Generalized modus ponens (GMP)

Premise 1: x is A′
Premise 2: IF x is A THEN y is B

Conclusion: y is B′

$$B' = A' \circ R(x, y)$$

$$\mu_{B'}(y) = \max_x \min(\mu_{A'}(x),\ \mu_R(x, y))$$

Forward: from an observed fact to a conclusion.

⊣ Generalized modus tollens (GMT)

Premise 1: y is B′
Premise 2: IF x is A THEN y is B

Conclusion: x is A′

$$A' = B' \circ R(x, y)$$

$$\mu_{A'}(x) = \max_y \min(\mu_{B'}(y),\ \mu_R(x, y))$$

Backward: from an observed effect to its likely cause.

Example 12: Generalized modus ponens

Given

$X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$ and the rule IF x is A THEN y is B (Zadeh max–min implication).

$A = \{(x_1, 0.5), (x_2, 1), (x_3, 0.6)\}$, $B = \{(y_1, 1), (y_2, 0.4)\}$

Observed fact: x is A′, with $A' = \{(x_1, 0.6), (x_2, 0.9), (x_3, 0.7)\}$.

Required

(a) Build the rule matrix $R = (A \times B) \cup (A^c \times Y)$.

(b) Infer $B' = A' \circ R$ by generalized modus ponens.

GMP, worked: from x is A′ to y is B′

$$A = \{(x_1, 0.5),\ (x_2, 1),\ (x_3, 0.6)\}, \quad B = \{(y_1, 1),\ (y_2, 0.4)\}$$

$$A' = \{(x_1, 0.6),\ (x_2, 0.9),\ (x_3, 0.7)\}$$

Rule matrix (Zadeh max–min)

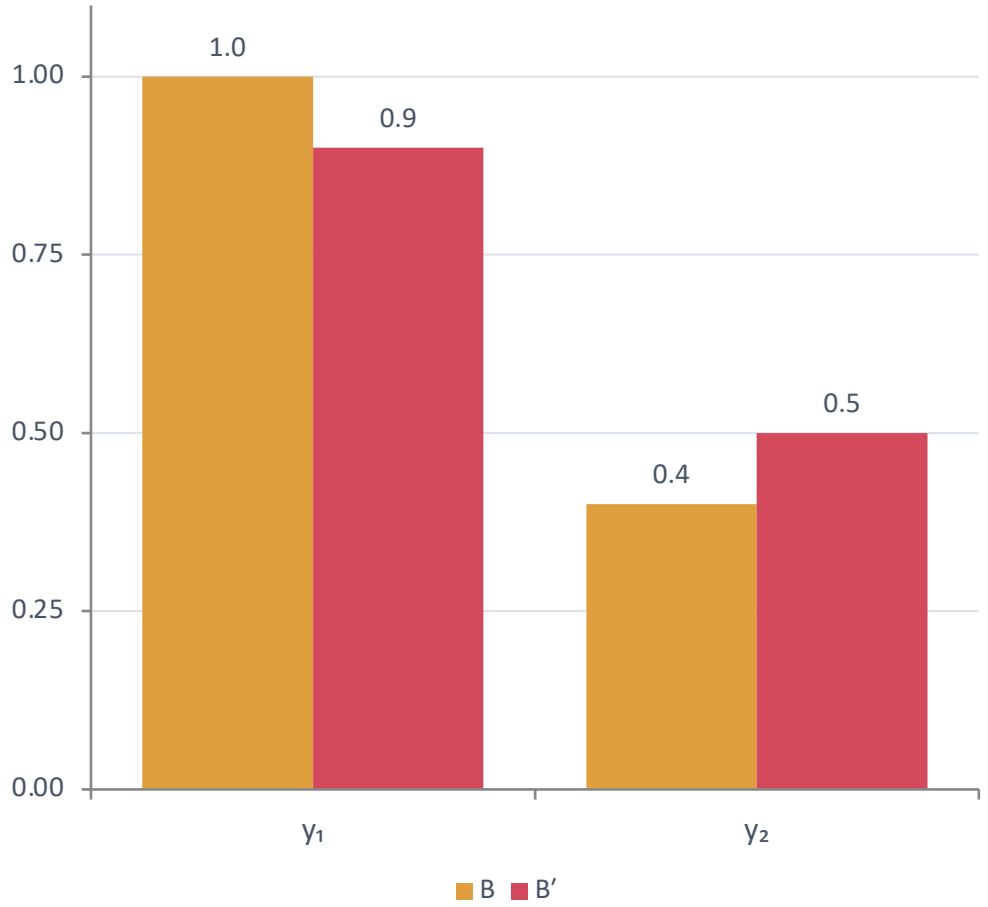
R	y_1	y_2
x_1	0.5	0.5
x_2	1	0.4
x_3	0.6	0.4

R = (A × B) ∪ (Ā × Y)

Rows x_1 … x_3, columns y_1, y_2

$$B' = [0.6 \quad 0.9 \quad 0.7] \circ \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} = [0.9 \quad 0.5]$$

Consequent B and inferred B′



Example 13: Generalized modus tollens

Given

The same rule as in the previous example: $A = \{(x_1, 0.5), (x_2, 1), (x_3, 0.6)\}$, $B = \{(y_1, 1), (y_2, 0.4)\}$, Zadeh max–min implication.

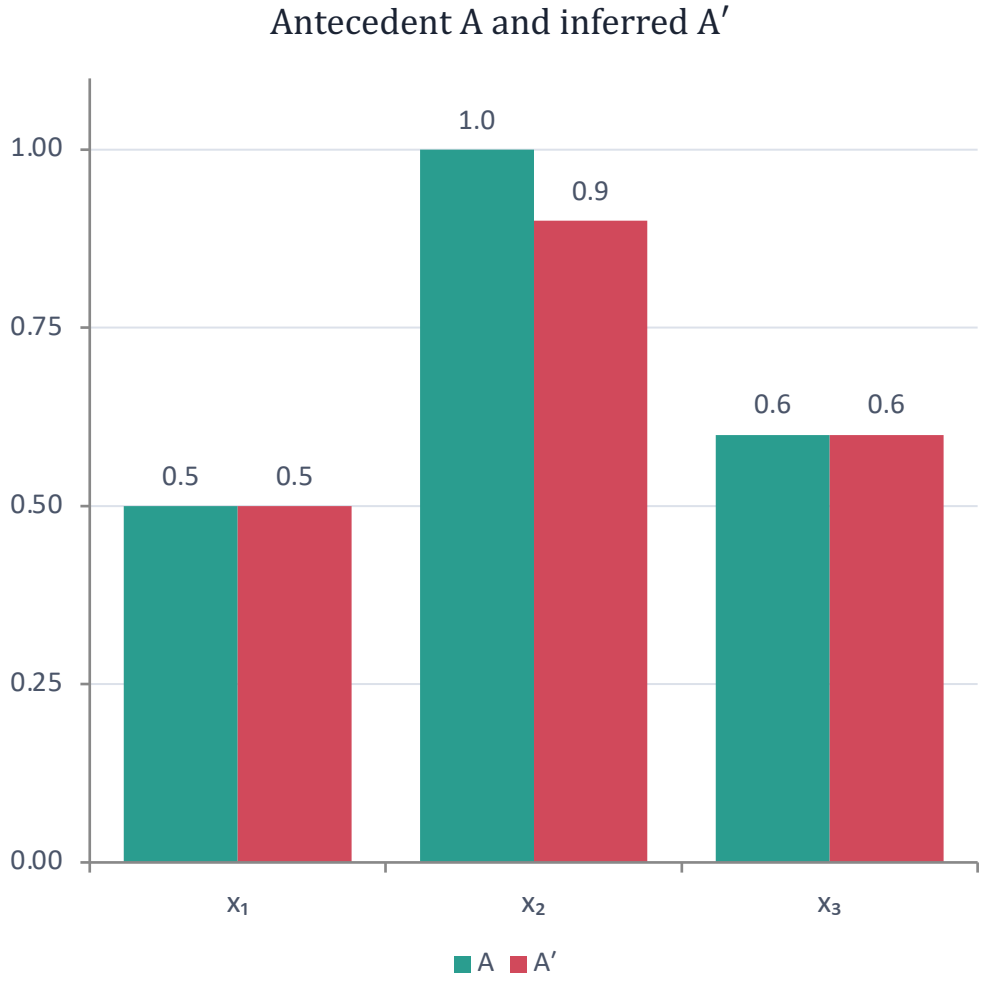
Observed fact: y is B′, with $B' = \{(y_1, 0.9), (y_2, 0.7)\}$.

Required

(a) Infer $A' = B' \circ R$ by generalized modus tollens (pair B′ with each row of R).

(b) Which value of x does the observation point back to most strongly?

GMT, worked: from y is B′ back to x is A′



$B' = \{(y_1, 0.9),\ (y_2, 0.7)\}, \qquad R$ as before

$$\mu_{A'}(x) = \max_y \min(\mu_{B'}(y),\ \mu_R(x, y))$$

$$\mu_{A'}(x_1) = \max\{\min(0.9,0.5),\ \min(0.7,0.5)\} = 0.5$$

$$\mu_{A'}(x_2) = \max\{\min(0.9,1),\ \min(0.7,0.4)\} = 0.9$$

$$\mu_{A'}(x_3) = \max\{\min(0.9,0.6),\ \min(0.7,0.4)\} = 0.6$$

$$A' = \{(x_1, 0.5),\ (x_2, 0.9),\ (x_3, 0.6)\}$$

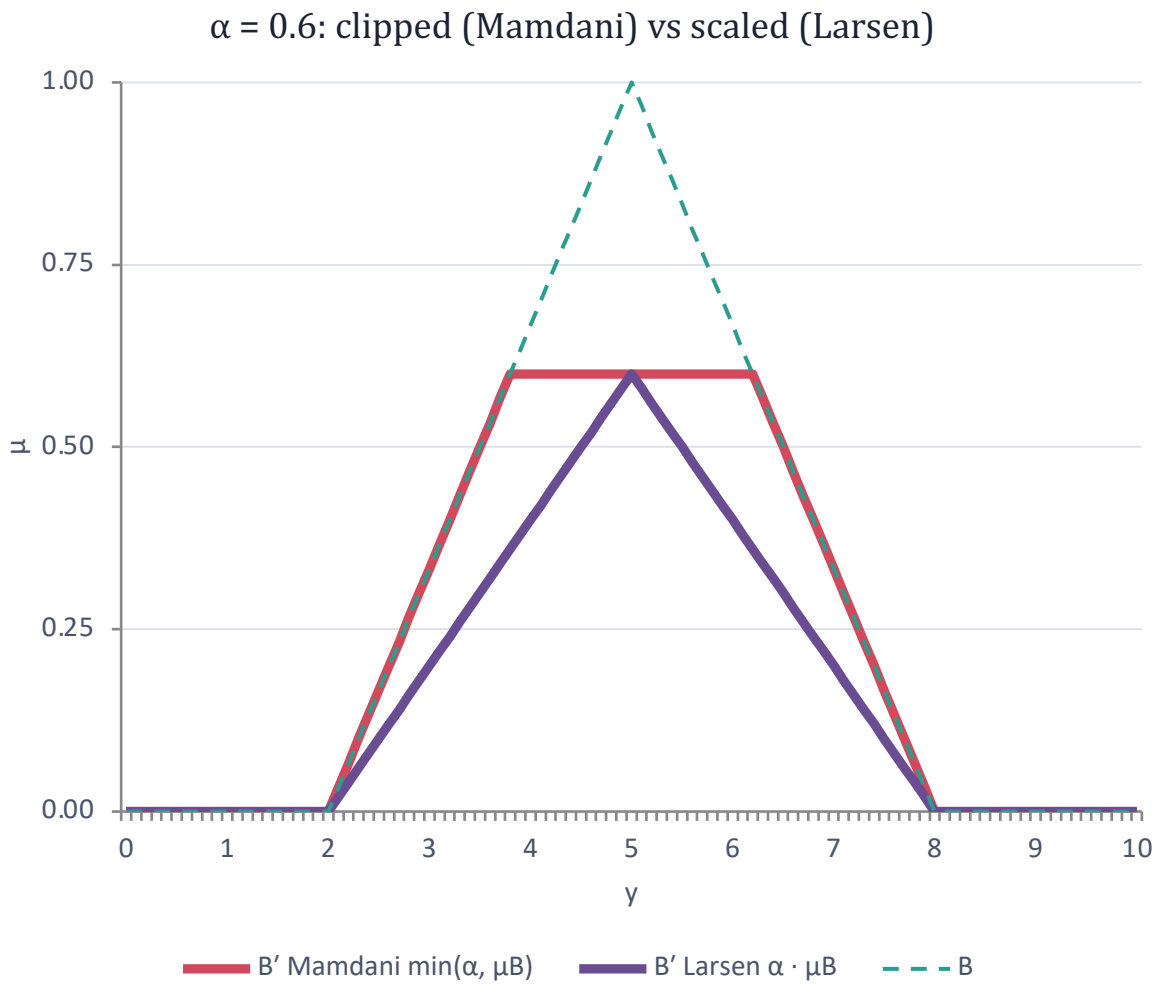
A crisp input turns inference into clipping

$$A' = \{(x_0, 1)\}, \qquad R = A \times B \ \text{(Mamdani)}$$

$$\mu_{B'}(y) = \min(\mu_A(x_0),\ \mu_B(y))$$

- A measured input is a fuzzy singleton A′.
- GMP then clips B at the firing strength α = μA(x_o).
- Larsen's product rule scales B by α instead.
- With several rules, take the union (max) of the clipped sets.

This is exactly what a Mamdani controller computes (sections 05 and 08).

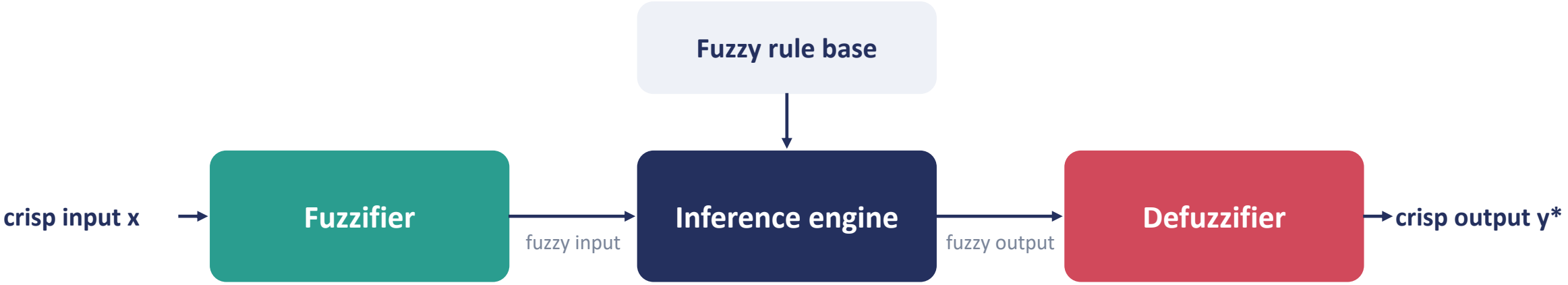


05

Defuzzification I

Why defuzzify · method families · λ-cuts · output of several rules

A fuzzy system speaks crisp at both ends



A controller needs a number

Rule: if temperature is HIGH then rotation is FAST. The motor needs a speed such as 30 rps, not the word “fast”.

? Which value is “high”?

T_{HIGH} = {(15, 0.1), (20, 0.4), …, (45, 0.85), (50, 0.9)}. The largest grade? The middle? Defuzzification answers with a rule, not a guess.

Defuzzification methods at a glance

λ Lambda-cut

Fuzzy set → crisp set, or fuzzy relation → crisp relation. Keep everything with $\mu \geq \lambda$.

Gives a set, not yet a single number

M Maxima methods

Height · first of maxima · last of maxima · mean of maxima

Fast; looks only at the peak

G Centroid methods

Centre of gravity · centre of sums · centre of largest area · bisector of area

Uses the whole shape; more computation

w Weighted average

Peak positions weighted by their grades (Sugeno style)

Fastest; symmetric output sets only

Example 14: Λ-cuts of fuzzy sets

Given

$A_1 = \{(x_1, 0.9), (x_2, 0.5), (x_3, 0.2), (x_4, 0.3)\}$

$A_2 = \{(x_1, 0.1), (x_2, 0.5), (x_3, 0.8), (x_4, 0.7)\}$

Required

(a) Find the λ-cut of A_1 for $\lambda = 0.6$.

(b) Find the λ-cut of A_2 for $\lambda = 0.2$.

(c) Is each result a fuzzy set or a crisp set? Is it a single crisp number?

λ-cut: keep elements whose grade reaches λ

$$A_\lambda = \{\, x \mid \mu_A(x) \geq \lambda \,\}, \qquad 0 \leq \lambda \leq 1$$

$$A_1 = \{(x_1, 0.9),\ (x_2, 0.5),\ (x_3, 0.2),\ (x_4, 0.3)\}$$

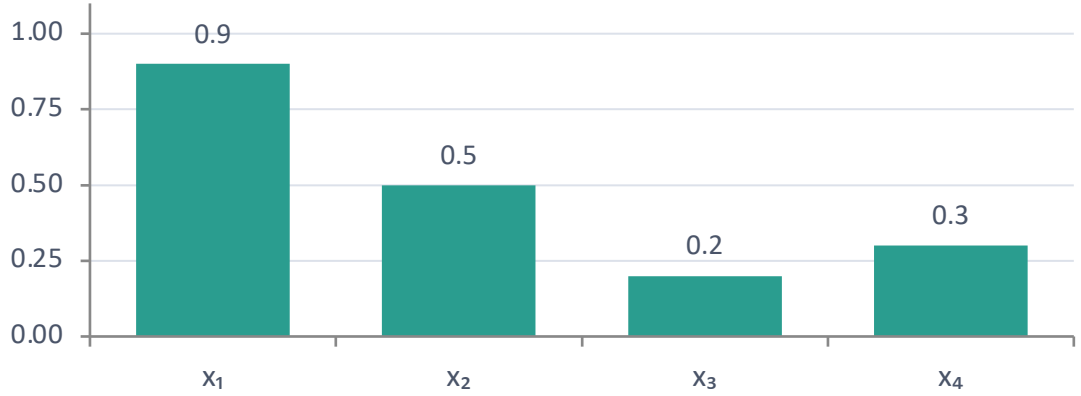
$$\lambda = 0.6: \quad A_{1,\,0.6} = \{x_1\}$$

$$A_2 = \{(x_1, 0.1),\ (x_2, 0.5),\ (x_3, 0.8),\ (x_4, 0.7)\}$$

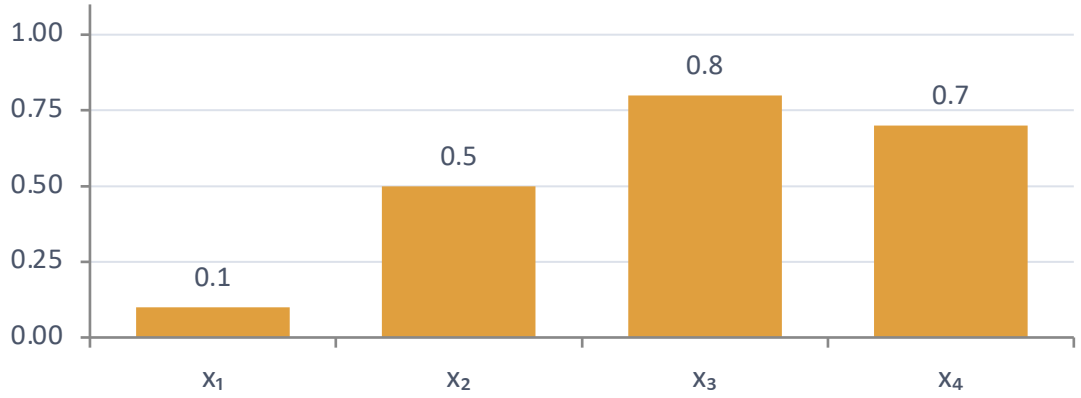
$$\lambda = 0.2: \quad A_{2,\,0.2} = \{x_2,\ x_3,\ x_4\}$$

- The result is a crisp set: it may be empty or hold several elements.
- To get one number, pick an element or average the survivors.
- It is the α-cut from lecture 3 with λ in place of α.

A_1: λ = 0.6 keeps only x_1



A_2: λ = 0.2 keeps x_2, x_3 and x_4



Example 15: Λ-cuts of a fuzzy relation

Given

R is a fuzzy relation on $X \times Y$, with $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$.

Required

(a) Find $R_{0.5}$.

(b) Find $R_{0.9}$.

(c) Find $R_{0.2}$ and explain the result.

R

	y_1	y_2	y_3
x_1	1	0.2	0.3
x_2	0.5	0.9	0.6
x_3	0.4	0.8	0.7

λ-cuts of relations, and their properties

R

	y_1	y_2	y_3
x_1	1	0.2	0.3
x_2	0.5	0.9	0.6
x_3	0.4	0.8	0.7

$R_{0.5}$ (entries ≥ 0.5)

	y_1	y_2	y_3
x_1	1	0	0
x_2	1	1	1
x_3	0	1	1

$R_{0.9}$ (entries ≥ 0.9)

	y_1	y_2	y_3
x_1	1	0	0
x_2	0	1	0
x_3	0	0	0

Properties (they also hold for relations R and S in place of A and B)

$$(A \cup B)_\lambda = A_\lambda \cup B_\lambda$$

$$(A \cap B)_\lambda = A_\lambda \cap B_\lambda$$

$$(A^c)_\lambda \neq (A_\lambda)^c \text{ in general}$$

$$\lambda \leq \alpha \Longrightarrow A_\alpha \subseteq A_\lambda, \quad A_0 = X$$

Example 16: Two rules fire at once

Given

Rule 1: IF x is A_1 THEN y is C_1. Rule 2: IF x is A_2 THEN y is C_2.

All four sets are triangles, written (left foot, peak, right foot):

$A_1 = (0, 2, 5)$, $A_2 = (2.3, 5.3, 8.3)$, $C_1 = (0, 3, 6)$, $C_2 = (4, 7, 10)$

Crisp input: $x' = 3.2$.

Required

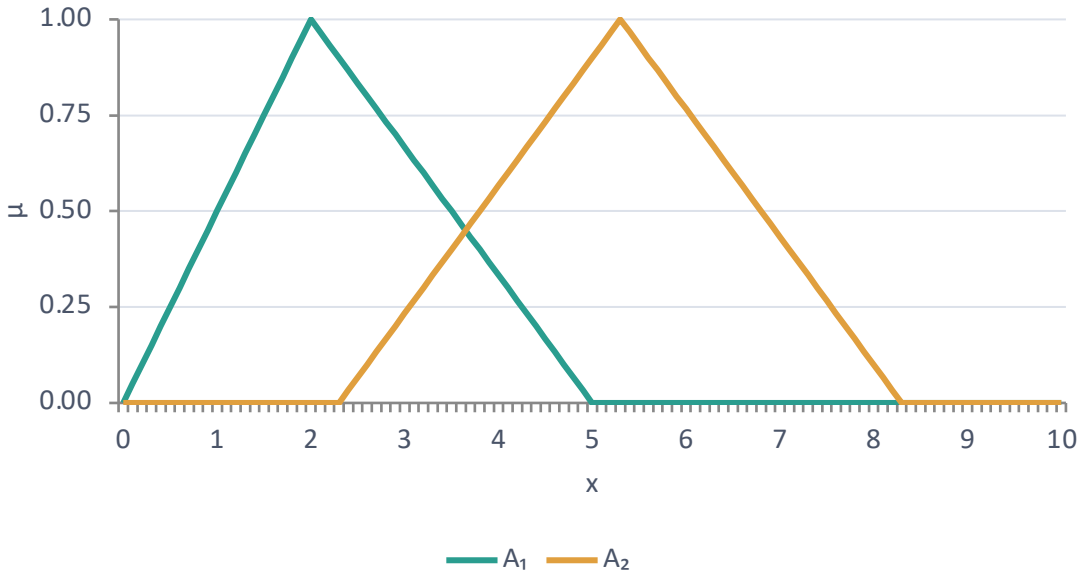
(a) Find the firing strengths $\alpha_1 = \mu_{A_1}(x')$ and $\alpha_2 = \mu_{A_2}(x')$.

(b) Clip each consequent at its firing strength.

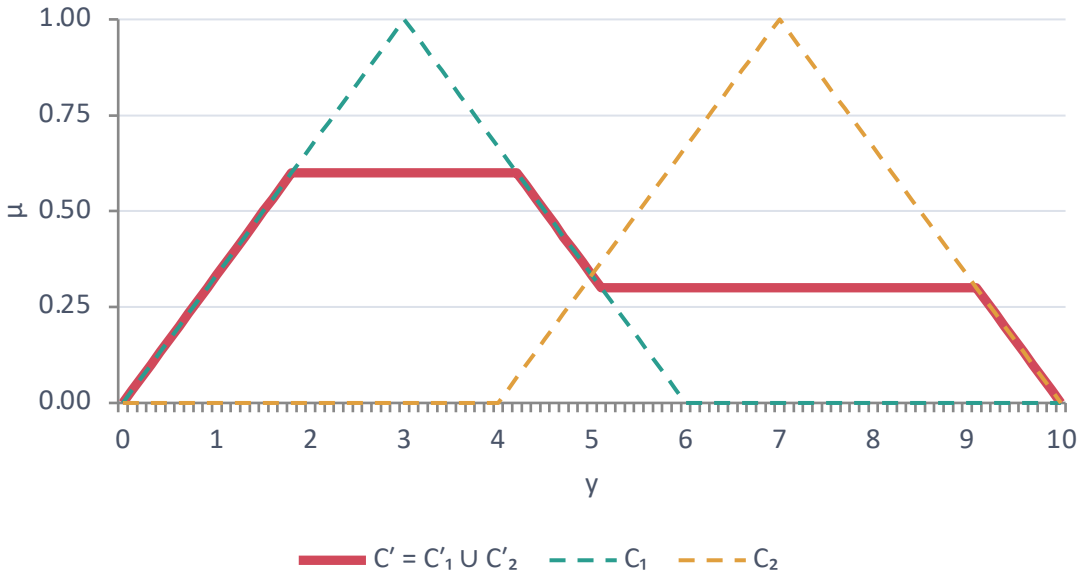
(c) Aggregate the clipped sets into the fuzzy output C'.

Several rules fire: clip, then take the union

Input: $x' = 3.2$ fires A_1 (0.6) and A_2 (0.3)



Output: clipped consequents and their union C'



1 Fuzzify

$\mu A_1(x') = 0.6, \ \mu A_2(x') = 0.3$

2 Clip each consequent

$C'_i(y) = \min(\alpha_i, C_i(y))$

3 Aggregate

$C' = C'_1 \cup C'_2$, the max of the clipped sets

06

Defuzzification II

Maxima methods · centre of gravity · centre of sums · bisector · weighted average

Example 17: Maxima methods

Given

The fuzzy output C′ of the two-rule example: $C_1 = (0, 3, 6)$ clipped at $\alpha_1 = 0.6$, $C_2 = (4, 7, 10)$ clipped at $\alpha_2 = 0.3$, joined by max.

Required

(a) Find the height h of C′ and the interval of y where C′ reaches it.

(b) Defuzzify C′ by first of maxima, last of maxima and mean of maxima.

(c) Can the height method be used here? Why or why not?

Maxima methods: height, first, last and mean

Height $\mu(x^*) \geq \mu(x) \quad \forall x$

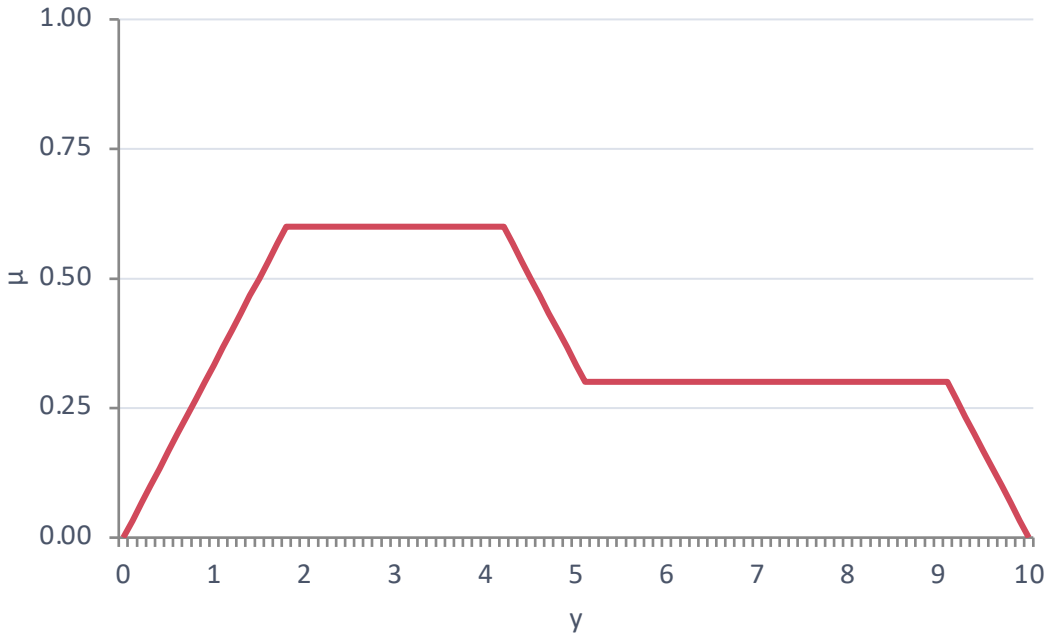
First of maxima $x^* = \min\{x \mid \mu(x) = h\}$

Last of maxima $x^* = \max\{x \mid \mu(x) = h\}$

Mean of maxima $x^* = \frac{\sum_{x_i \in M} x_i}{|M|}, \quad M = \{x_i \mid \mu(x_i) = h\}$

h = height of the set. The height method needs a unique peak.

C′ has a plateau at 0.6 from 1.8 to 4.2



C′: FoM = 1.8, LoM = 4.2, MoM = 3.0

Young example: FoM = 20, LoM = 25, MoM = 22.5

Example 18: Defuzzifying a discrete set

Given

The aggregated output B is a discrete fuzzy set (Raol & George, eq. 5.30):

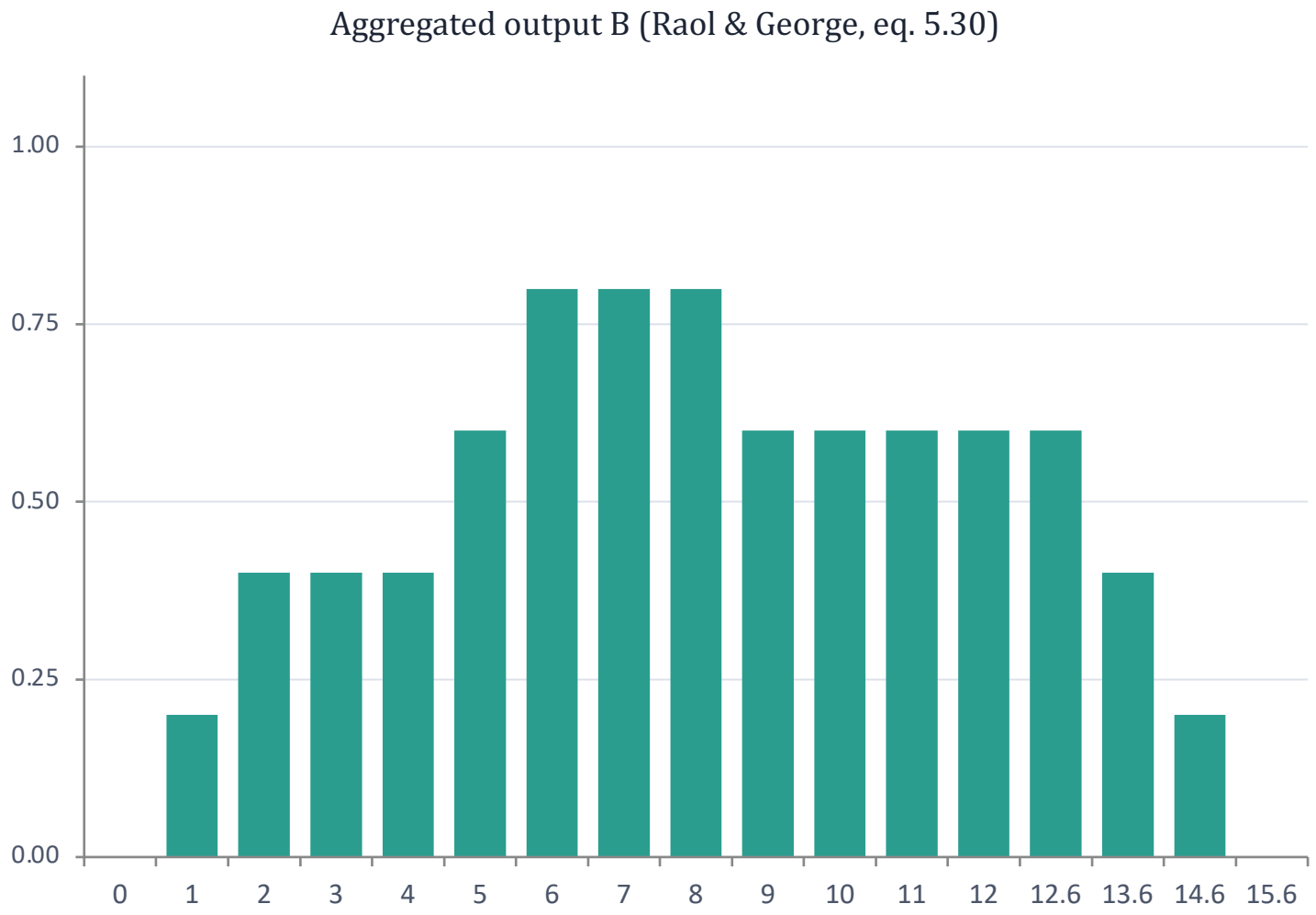
v	0	1	2	3	4	5	6	7	8	9	10	11	12	12.6	13.6	14.6	15.6
μ	0	0.2	0.4	0.4	0.4	0.6	0.8	0.8	0.8	0.6	0.6	0.6	0.6	0.6	0.4	0.2	0

Required

(a) Defuzzify B by centre of gravity, $x^* = \Sigma v\mu / \Sigma \mu$.

(b) Defuzzify B by mean of maxima, first of maxima and last of maxima.

One discrete set, four crisp answers



8.09

Centre of gravity

Σ νμ / Σ μ

7

Mean of maxima

mean of 6, 7, 8

6

First (smallest) of maxima

8

Last (largest) of maxima

CoG worked: why multiply v by μ?

μ	Points v with this μ	Adds to Σ μ	Adds to Σ vμ
0.2	1, 14.6	2 × 0.2 = 0.4	0.2 × 15.6 = 3.12
0.4	2, 3, 4, 13.6	4 × 0.4 = 1.6	0.4 × 22.6 = 9.04
0.6	5, 9, 10, 11, 12, 12.6	6 × 0.6 = 3.6	0.6 × 59.6 = 35.76
0.8	6, 7, 8	3 × 0.8 = 2.4	0.8 × 21 = 16.8
Total	**15 points with μ > 0**	**Σ μ = 8**	**Σ vμ = 64.72**

Group equal grades: μ × (sum of their v). Points with μ = 0 (v = 0 and 15.6) add nothing.

$$x^* = \frac{\sum v\,\mu}{\sum \mu} = \frac{64.72}{8} = 8.09$$

Why multiply by μ?

- CoG is a weighted average: each v counts only as much as it belongs to B, so μ is its weight.
- Picture the v axis as a beam with weight μ hung at each v. x* is where the beam balances: Σ position × mass / Σ mass.
- Dividing by Σ μ turns the weighted total back into a position on the v axis.
- Check: v = 0 (μ = 1) and v = 10 (μ = 0.25) give 2.5 / 1.25 = 2, pulled toward the stronger point.

Raol & George §5.3.5, eq. (5.32), p. 183

Centre of gravity (CoG)

$$x^* = \frac{\int x\,\mu_C(x)\,dx}{\int \mu_C(x)\,dx}$$

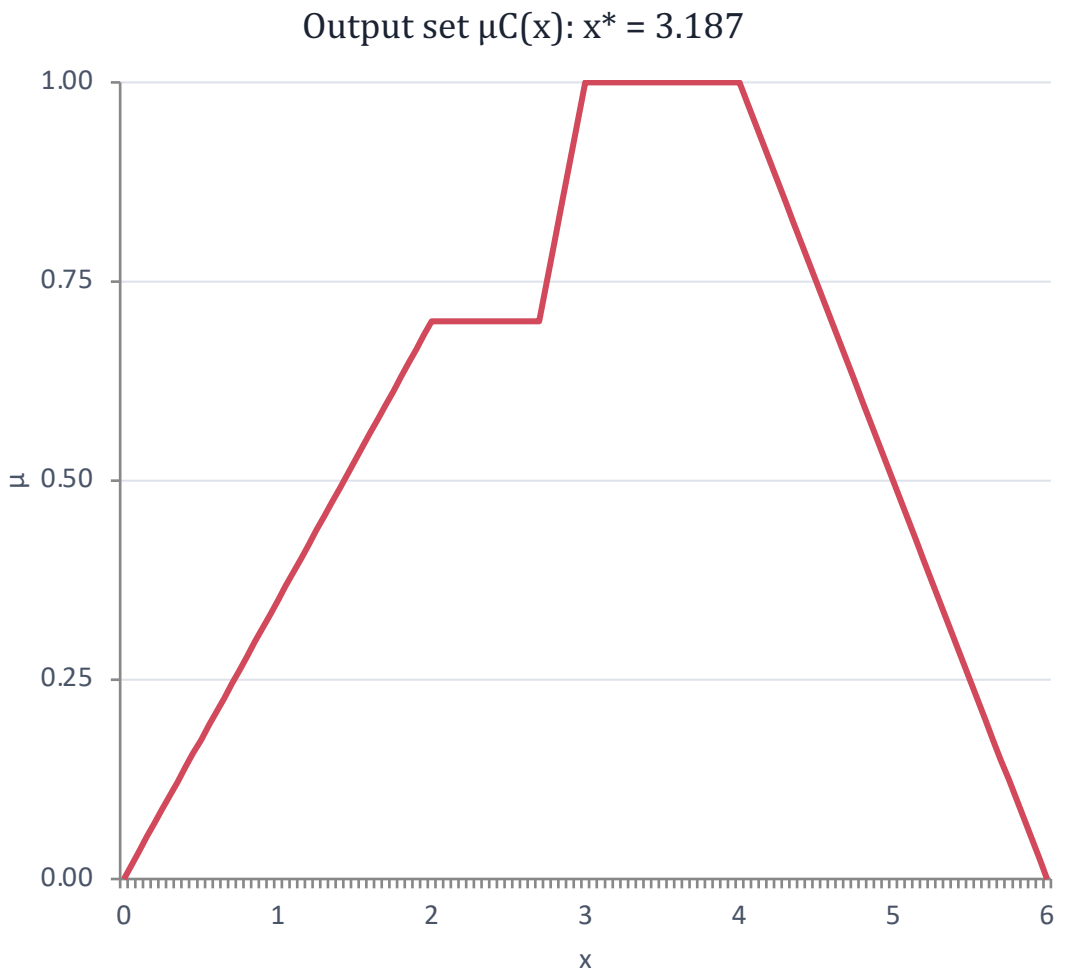
Discrete universe

$$x^* = \frac{\sum_{i=1}^{n} x_i\ \mu_C(x_i)}{\sum_{i=1}^{n} \mu_C\,(x_i)}$$

By hand: split the shape into simple pieces

$$x^* = \frac{\sum_i A_i\ \bar{x}_i}{\sum_i A_i}$$

A_i = area of piece i (triangle, rectangle, trapezium), $\bar{x}_i$ = its centroid



Example 19: Centre of gravity by hand

Given

The aggregated output C has the piecewise-linear membership function

$\mu_C(x) = 0.35x$ for $0 \le x < 2$

$\mu_C(x) = 0.7$ for $2 \le x < 2.7$

$\mu_C(x) = x - 2$ for $2.7 \le x < 3$

$\mu_C(x) = 1$ for $3 \le x < 4$

$\mu_C(x) = -0.5x + 3$ for $4 \le x \le 6$

Required

(a) Find the area $\int \mu\, dx$ and the moment $\int x\mu\, dx$ of each of the five pieces.

(b) Find the crisp output x^* by centre of gravity.

CoG, worked: area and moment piece by piece

$$\mu_C(x) = \begin{cases} 0.35x & 0 \le x < 2 \\ 0.7 & 2 \le x < 2.7 \\ x - 2 & 2.7 \le x < 3 \\ 1 & 3 \le x < 4 \\ -0.5x + 3 & 4 \le x \le 6 \end{cases}$$

Each line comes from two points it passes through. A_5 passes (4, 1) and (6, 0), so y − 1 = −½(x − 4), that is y = −0.5x + 3.

Piece	Interval	Area ∫μ dx	Moment ∫xμ dx
A_1	0 ≤ x < 2	0.700	0.933
A_2	2 ≤ x < 2.7	0.490	1.151
A_3	2.7 ≤ x < 3	0.255	0.729
A_4	3 ≤ x < 4	1.000	3.500
A_5	4 ≤ x ≤ 6	1.000	4.667
Total		**D = 3.445**	**N = 10.980**

$$x^* = \frac{N}{D} = \frac{10.98}{3.445} = 3.187$$

Example 20: Centre of sums

Given

Three clipped output sets, each a trapezoid written (left foot, left shoulder, right shoulder, right foot) with its height:

$C_1 = (0, 1, 4, 5)$ at height 0.3

$C_2 = (3, 4, 6, 7)$ at height 0.5

$C_3 = (5, 6, 7, 8)$ at height 1.0

Required

(a) Find the area and the centre of each set.

(b) Defuzzify by centre of sums.

(c) Defuzzify by centre of largest area, and compare both with the centre of gravity of the union.

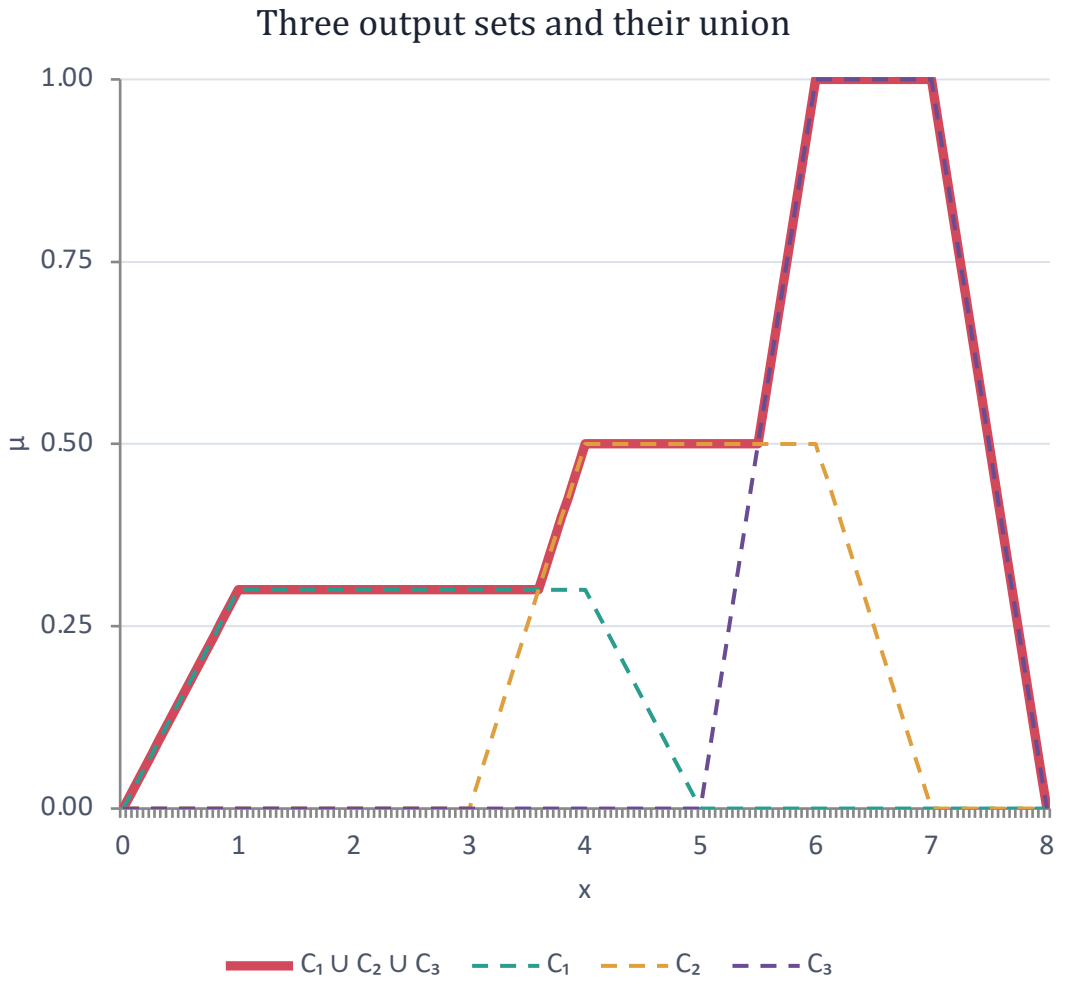
Centre of sums and centre of largest area

$$x^* = \frac{\sum_{i=1}^{n} \bar{x}_i \; A_{C_i}}{\sum_{i=1}^{n} A_{C_i}}$$

Each output set is used on its own, so overlapping areas count twice. No union is needed.

Set	Area A	Centre x̄
C_1	½ × 0.3 × (3 + 5) = 1.2	2.5
C_2	½ × 0.5 × (4 + 2) = 1.5	5
C_3	½ × 1.0 × (3 + 1) = 2	6.5

CoS = 5 · CoG of the union = 4.95 · largest area (C_3) = 6.5



Example 21: Bisector of area

Given

The fuzzy output C′ of the two-rule example: $C_1 = (0, 3, 6)$ clipped at $\alpha_1 = 0.6$, $C_2 = (4, 7, 10)$ clipped at $\alpha_2 = 0.3$, joined by max (triangles written as left foot, peak, right foot).

Required

(a) Find the area of C′. Hint: add the two clipped areas and subtract their overlap.

(b) Find the value x* that splits the area of C′ into two equal halves.

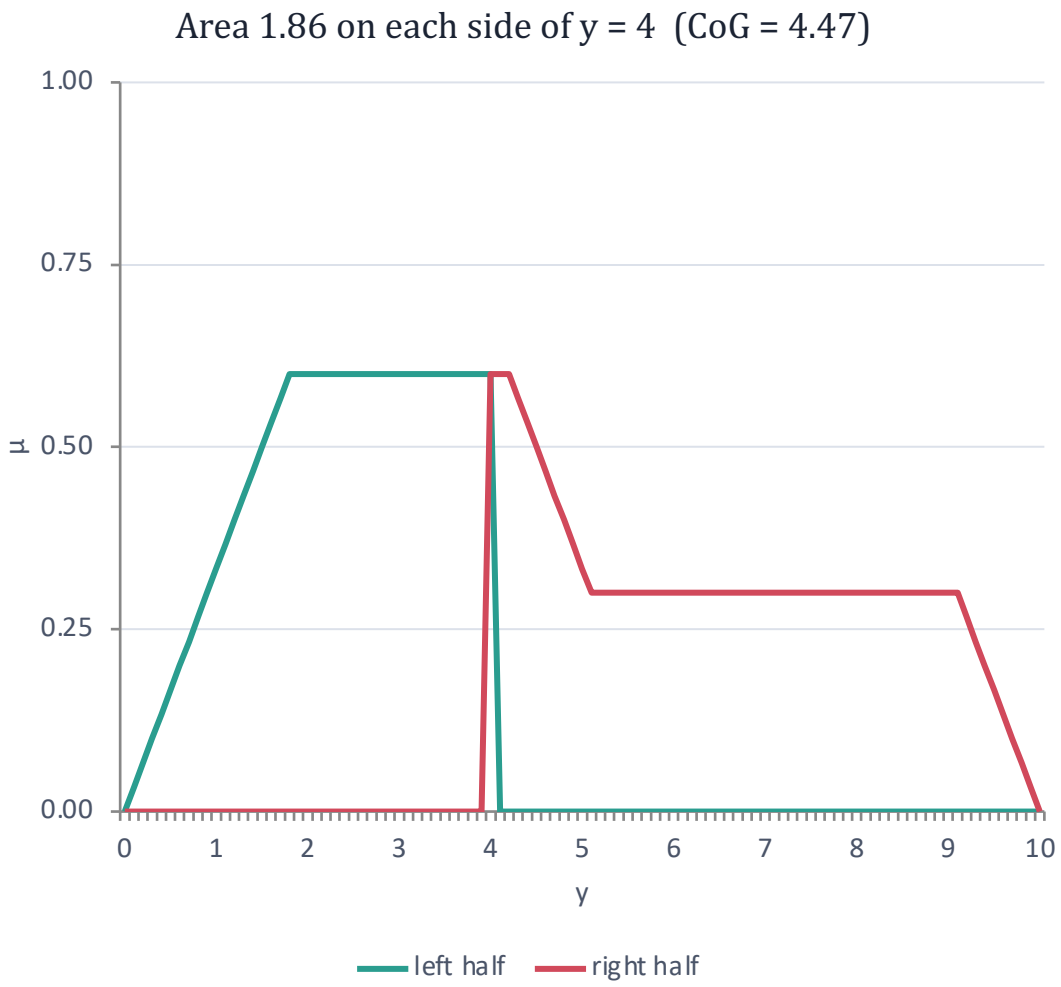
(c) Compare x* with the centre of gravity of C′. Why do they differ?

Bisector of area: split the area in half

$$\int_a^{x^*} \mu_C(x)\,dx = \int_{x^*}^{b} \mu_C(x)\,dx$$

Two-rule C′	Area
$C'_1 + C'_2$	2.52 + 1.53 = 4.05
minus their overlap	− 0.33
$C' = C'_1 \cup C'_2$	**3.72, half = 1.86**
Left ramp, 0 to 1.8	½ × 1.8 × 0.6 = 0.54
Plateau, 1.8 to x*	0.6 (x* − 1.8) = 1.86 − 0.54

$$x^* = 1.8 + \frac{1.86 - 0.54}{0.6} = 4$$



Example 22: Weighted average

Given

The two-rule example: $C_1 = (0, 3, 6)$ clipped at $\alpha_1 = 0.6$ and $C_2 = (4, 7, 10)$ clipped at $\alpha_2 = 0.3$ (triangles written as left foot, peak, right foot).

Required

(a) Why may the weighted-average method be used for these sets?

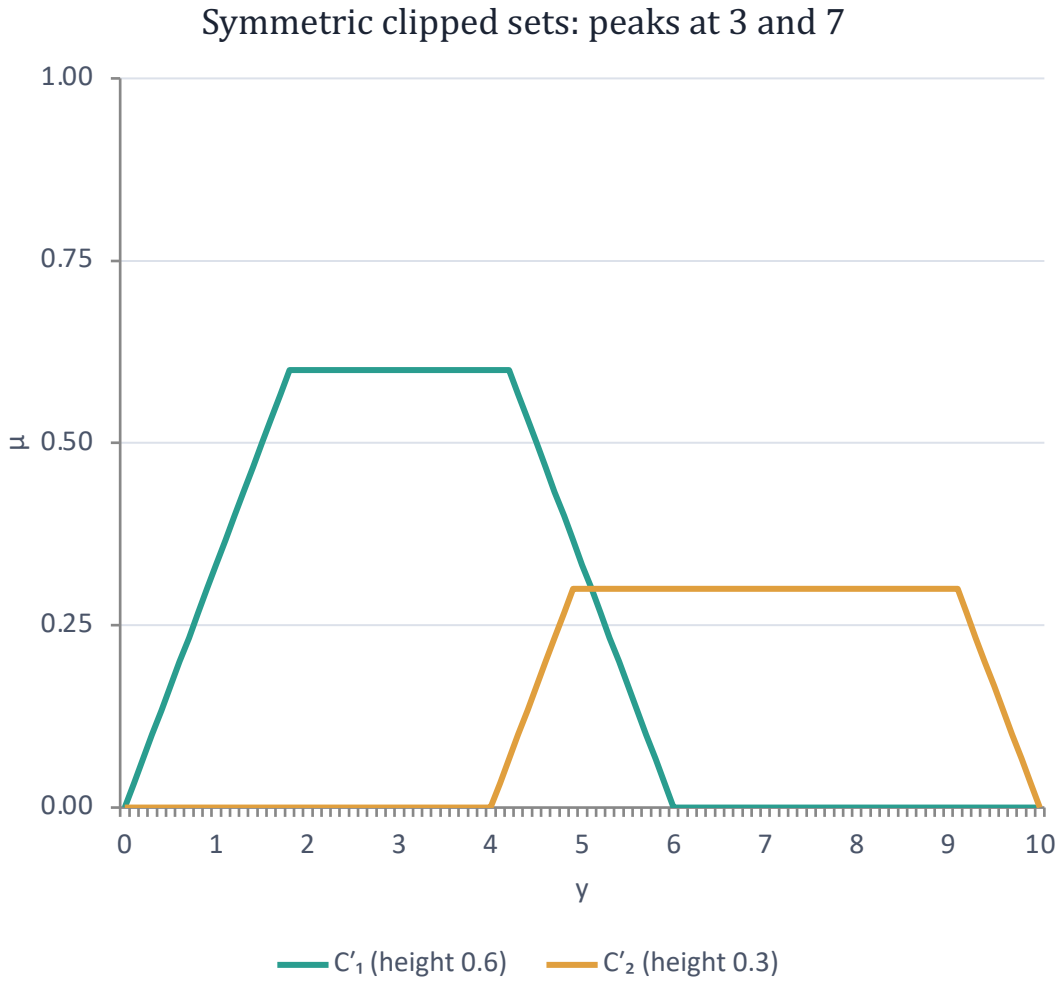
(b) Find the crisp output x* by the weighted-average method.

Weighted average: peaks weighted by heights

$$x^* = \frac{\sum_i \mu_{C_i}(\bar{x}_i)\,\bar{x}_i}{\sum_i \mu_{C_i}(\bar{x}_i)}$$

- $\bar{x}_i$: centre (peak) of output set i; μ: its clipped height.
- Valid only for symmetric output sets, where the peak is also the centroid.
- No integration at all: the cheapest centroid-like method.
- Takagi–Sugeno controllers use exactly this form (section 08).

$$x^* = \frac{0.6 \cdot 3 + 0.3 \cdot 7}{0.6 + 0.3} = 4.33$$



Same fuzzy output, seven crisp answers

Method	Uses	Cost	Watch out
Maxima (FoM, LoM, MoM)	the peak only	lowest	ignores the shape
Centre of gravity	the union	highest (integral)	the standard choice
Bisector of area	the union	high (integral)	balances area, not moment
Centre of sums	each set	medium	overlaps count twice
Centre of largest area	one set	low	small sets ignored
Weighted average	peaks and heights	lowest	symmetric sets only

All answers are acceptable. Choose by the accuracy and speed the application needs.



07

Fuzzy logic controller I

Why fuzzy control · FLC architecture · Mamdani vs Takagi–Sugeno · robot rule base

When a fuzzy controller is the right tool

~ Nonlinear

The input–output relation is not a straight line and has no clean equation.

t Time-varying

The system changes as it runs, so a fixed model goes stale.

± Noisy

Measurements are disturbed by the environment and never exact.

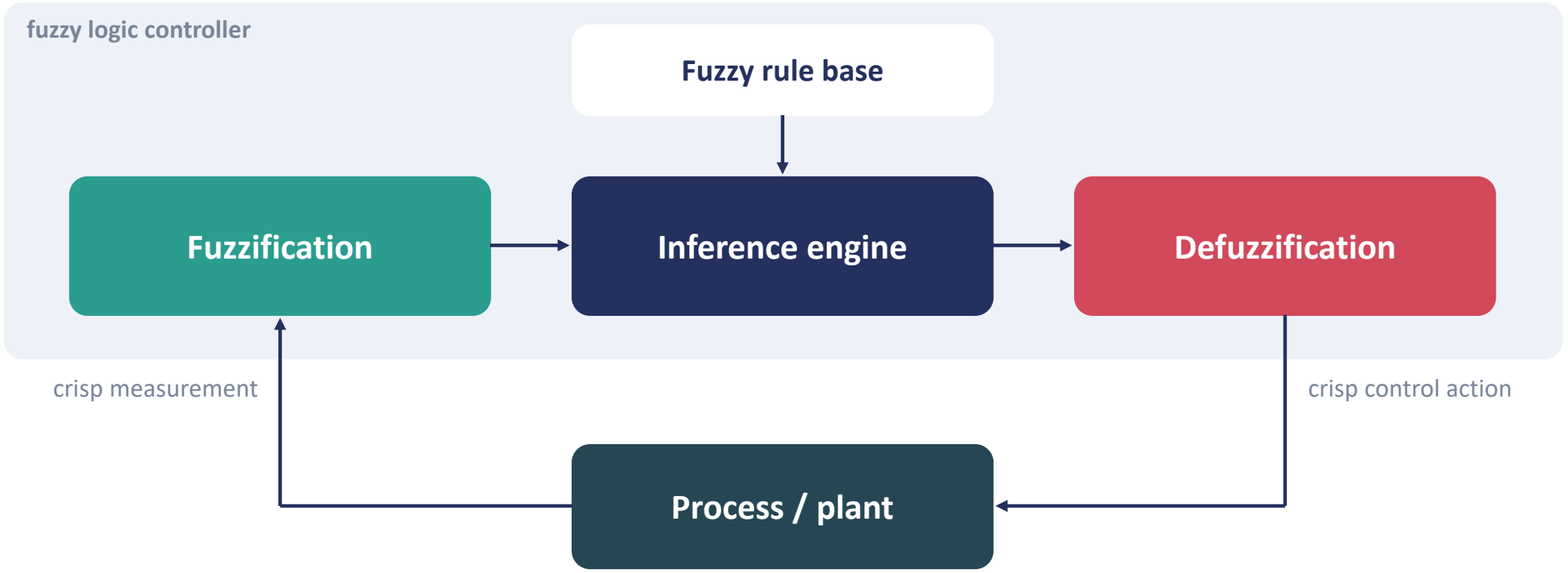
An FLC is a small expert system

- Knowledge base: IF–THEN rules written with a human expert.
- Inference engine: evaluates the rules on the current input.
- Fuzzy reasoning (control) is the most used application of fuzzy logic.

Typical applications

- Air conditioners and washing machines
- Camera autofocus and image stabilisation
- Automatic gearboxes, anti-lock brakes, cruise control
- Mobile-robot navigation (our case study)

Inside a fuzzy logic controller



1. Measure the inputs (crisp values)
2. Fuzzify: a grade for each linguistic term
3. Infer: fire the rules, combine their outputs
4. Defuzzify: one crisp action, then repeat

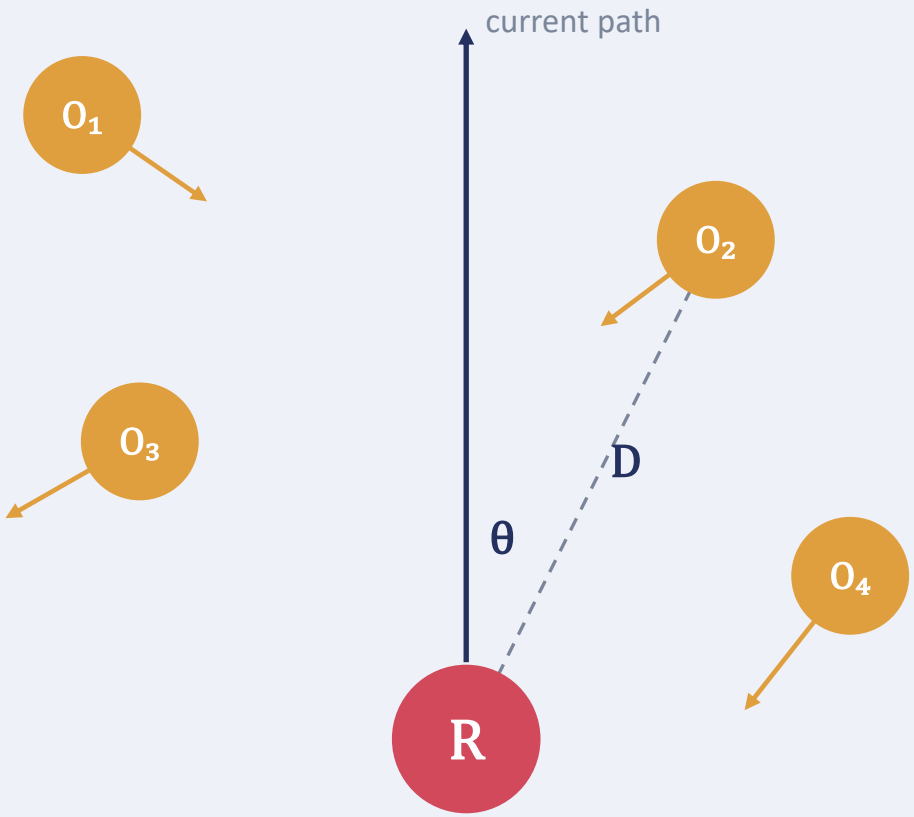
Two styles of FLC: Mamdani and Takagi–Sugeno

	Mamdani	Takagi–Sugeno
Rule consequent	a fuzzy set: δ is RT	a function: $y = a_o + a_1x_1 + \ldots + a_nx_n$
Modelling style	linguistic	precise, numerical
Interpretability	high: anyone can read the rules	low: the coefficients carry the meaning
Accuracy	lower	higher
Output step	aggregate, then defuzzify (CoG, CoS)	weighted average, no defuzzifier
Speed	slower	faster

Mamdani rule: IF D is NR AND θ is AA THEN δ is RT

Takagi–Sugeno rule: IF I_1 is L AND I_2 is FR THEN $y = I_1 + 2I_2$

Case study: steering a mobile robot



D **Input 1: distance to the obstacle**

$D \in [0.1, 2.2]$ m

θ **Input 2: angle of the obstacle relative to the robot**

$\theta \in [-90°, 90°]$

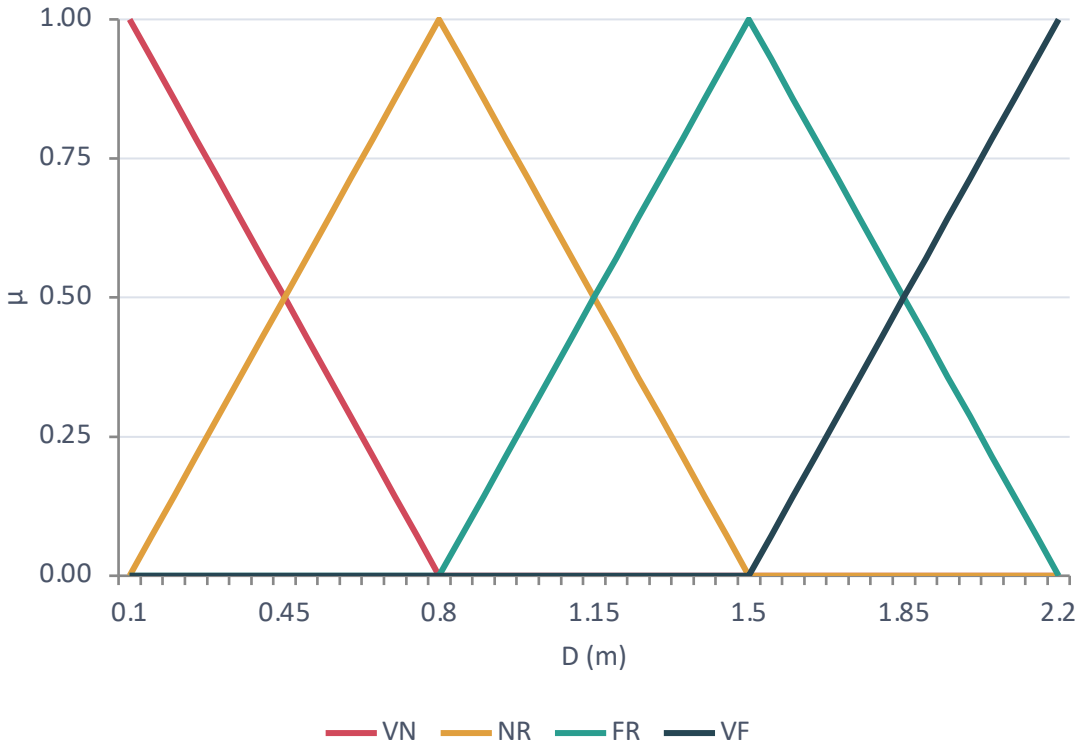
δ **Output: deviation from the current path**

$\delta \in [-90°, 90°]$, left negative, right positive

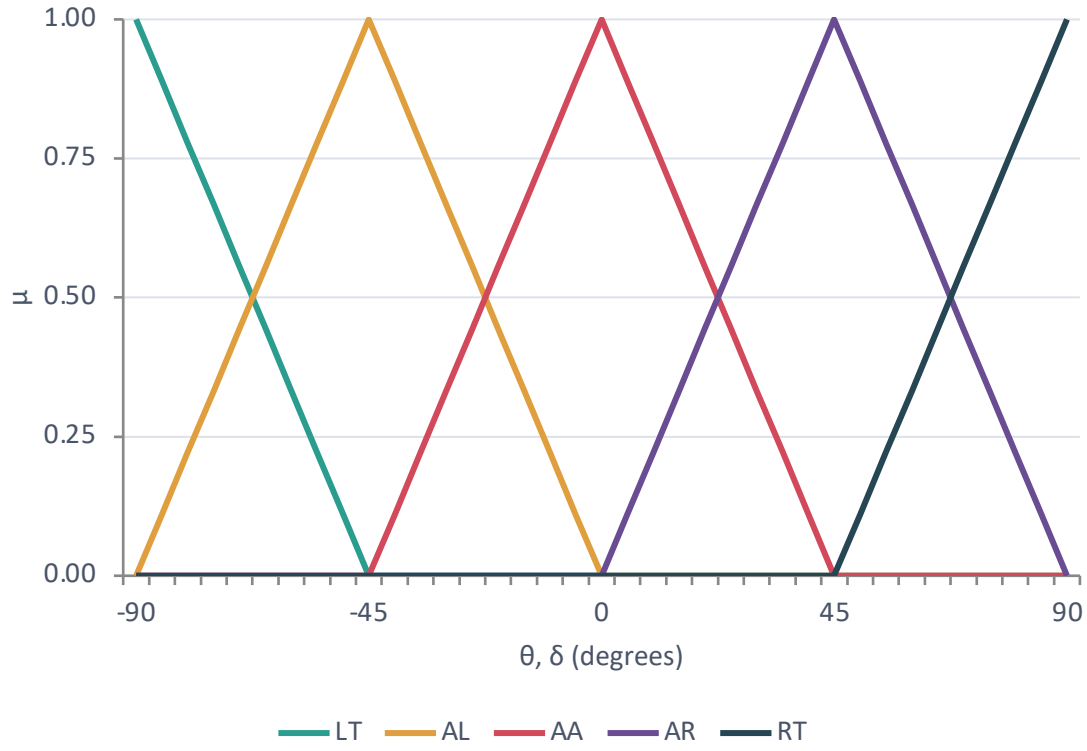
Assumptions: four moving obstacles of equal size and speed; at every instant the controller decides δ.

Linguistic terms and membership functions

Distance D: VN, NR, FR, VF



Angle θ and deviation δ: LT, AL, AA, AR, RT



VN very near · NR near · FR far · VF very far. LT left · AL ahead-left · AA ahead · AR ahead-right · RT right. θ and δ share the same five sets.

Rule base: 4 × 5 = 20 rules

D \ θ	LT	AL	AA	AR	RT
VN	AA	AR	AL	AL	AA
NR	AA	AA	RT	AA	AA
FR	AA	AA	AR	AA	AA
VF	AA	AA	AA	AA	AA

δ: AL ahead-left · AA ahead · AR ahead-right · RT right

Reading the matrix

IF D is VN AND θ is AA THEN δ is AL

- Obstacle very near and straight ahead: swerve ahead-left.
- Very near and ahead-left: deviate ahead-right.
- Far or very far: keep going (AA) almost everywhere.
- 4 distance terms × 5 angle terms = 20 rules.

08

Fuzzy logic controller II

Mamdani inference for the robot · Takagi–Sugeno inference

Example 23: Steering the mobile robot

Given

Inputs: distance $D \in [0.1, 2.2]$ m and angle $\theta \in [-90°, 90°]$ of the obstacle. Output: deviation $\delta \in [-90°, 90°]$ (right is positive).

D: triangles VN, NR, FR, VF with peaks at 0.1, 0.8, 1.5, 2.2 m.

θ and δ: triangles LT, AL, AA, AR, RT with peaks at −90°, −45°, 0°, 45°, 90°.

Each triangle's feet lie on its neighbours' peaks. Use the 20-rule base of the case study, min for AND and implication, max for aggregation.

Measured: $D = 1.04$ m, $\theta = 30°$.

Required

(a) Fuzzify D and θ.

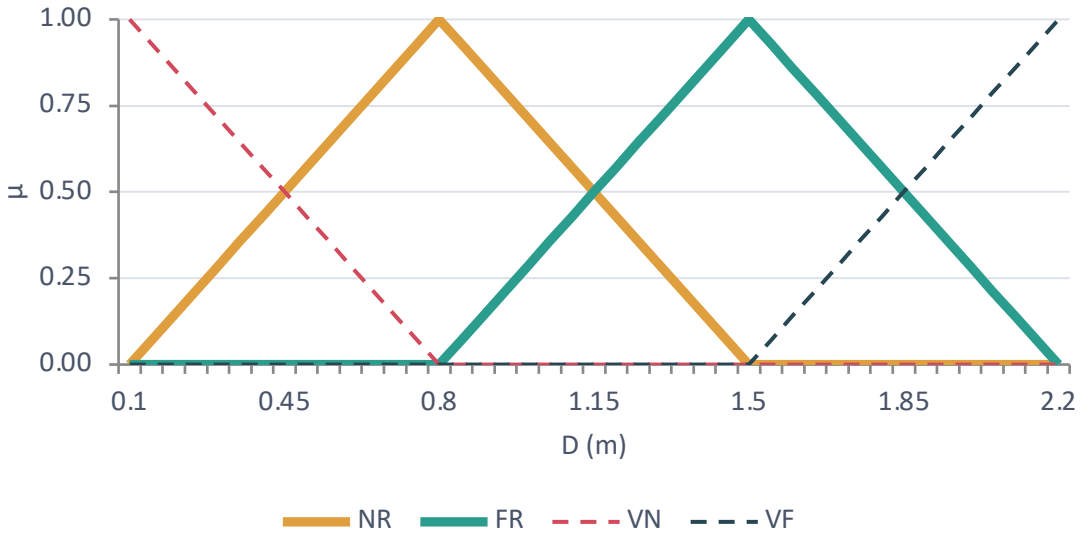
(b) Find the rules that fire and their strengths α.

(c) Clip each consequent and aggregate the results.

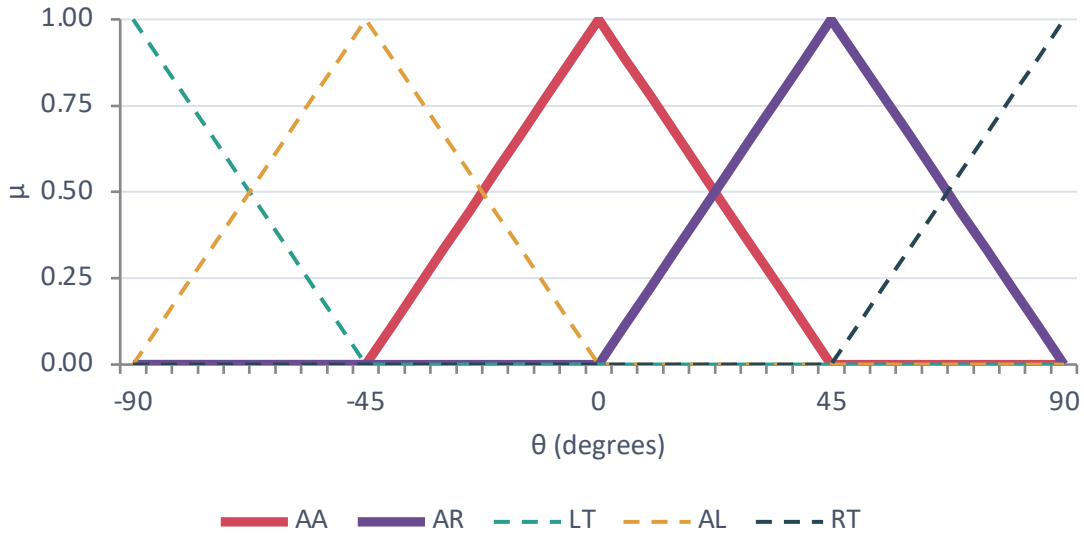
(d) Defuzzify by centre of sums: how far, and to which side, should the robot steer?

Step 1 — fuzzify D = 1.04 m and θ = 30°

D = 1.04 m touches NR and FR



θ = 30° touches AA and AR



$$\mu_{NR}(1.04) = \frac{1.5 - 1.04}{1.5 - 0.8} = 0.6571$$

$$\mu_{FR}(1.04) = \frac{1.04 - 0.8}{1.5 - 0.8} = 0.3429$$

$$\mu_{AA}(30) = \frac{45 - 30}{45 - 0} = 0.3333$$

$$\mu_{AR}(30) = \frac{30 - 0}{45 - 0} = 0.6667$$

Step 2 — fire the rules: strength α = min

D \ θ	LT	AL	AA	AR	RT
VN	AA	AR	AL	AL	AA
NR	AA	AA	RT	AA	AA
FR	AA	AA	AR	AA	AA
VF	AA	AA	AA	AA	AA

Rule	D term (μ)	θ term (μ)	δ	α = min
R1	NR 0.6571	AA 0.3333	RT	0.3333
R2	NR 0.6571	AR 0.6667	AA	0.6571
R3	FR 0.3429	AA 0.3333	AR	0.3333
R4	FR 0.3429	AR 0.6667	AA	0.3429

Only rows NR, FR and columns AA, AR are active: 4 of the 20 rules fire.

$$\alpha(R_i) = \min\left(\mu_{D_i}(1.04),\ \mu_{\theta_i}(30^\circ)\right)$$

Optional threshold: keeping only α ≥ 0.34 leaves R2 and R4, both "δ is AA". We keep all four rules in the final calculation.

Step 3 — clip each consequent and aggregate

Implication: min (clip)

- R1: RT clipped at 0.333
- R2: AA clipped at 0.657
- R3: AR clipped at 0.333
- R4: AA clipped at 0.343

Aggregation: max (union)

R2 and R4 both clip AA; the union keeps the higher cut. The result is one fuzzy set for δ, ready to defuzzify.



Step 4 — defuzzify: steer about 20° right

19.6°

deviation δ* by centre of sums: turn about 20° to the right of the current path

CoG of the union: 19.4°
The two centroid methods agree to within a quarter of a degree.

With rounded areas, this area-weighted sum gives 19.59°.

Rule	Clipped set	Area A	Centroid x̄	A × x̄
R1	RT at 0.333	12.50	71.0	887.5
R2	AA at 0.657	39.71	0.0	0.0
R3	AR at 0.333	25.00	45.0	1125.0
R4	AA at 0.343	25.57	0.0	0.0
Total		**102.78**		**2012.5**

$$\delta^* = \frac{\sum A_i \, \bar{x}_i}{\sum A_i} = \frac{2012.5}{102.78} = 19.58^\circ$$

Takagi–Sugeno: rules with formula consequents

Rule i: IF x_1 is A_1^i AND … AND x_n is A_n^i THEN

1

$$y^i = a_0^i + a_1^i x_1 + a_2^i x_2 + \cdots + a_n^i x_n$$

Weight of rule i: the product of its grades

2

$$w^i = \mu_{A_1^i}(x_1) \times \mu_{A_2^i}(x_2) \times \cdots \times \mu_{A_n^i}(x_n)$$

Crisp output directly, no defuzzifier

3

$$y = \frac{\sum_{i=1}^{k} w^i \; y^i}{\sum_{i=1}^{k} w^i}$$

What the designer supplies

- Membership functions for every input term.
- Coefficients a_o … a_n for every rule: the hard part, set from expertise or fitted to data.
- No output sets and no defuzzifier: the weighted average is already crisp.
- With constant consequents the output is exactly the weighted-average method.

Example 24: A Takagi–Sugeno controller

Given

Two inputs: I_1 with terms L, M, H and I_2 with terms NR, FR, VF. Nine rules of the form

IF I_1 is A AND I_2 is B THEN $y = a \cdot I_1 + b \cdot I_2$, with a = 1, 2, 3 for L, M, H and b = 1, 2, 3 for NR, FR, VF.

I_1: L = 1 up to 5, falling to 0 at 10; M = triangle (5, 10, 15); H rising from 0 at 10 to 1 at 15.

I_2: NR = 1 up to 1, falling to 0 at 2; FR = triangle (1, 2, 3); VF rising from 0 at 2 to 1 at 3.

Inputs: $I_1 = 6.0$, $I_2 = 2.2$.

Required

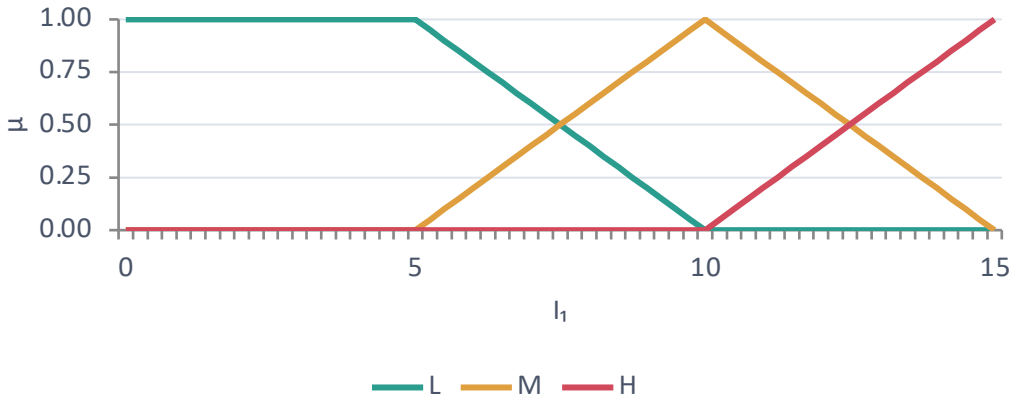
(a) Find the membership grades of I_1 and I_2.

(b) For each rule that fires, find its weight $w = \mu \times \mu$ and its output y^i.

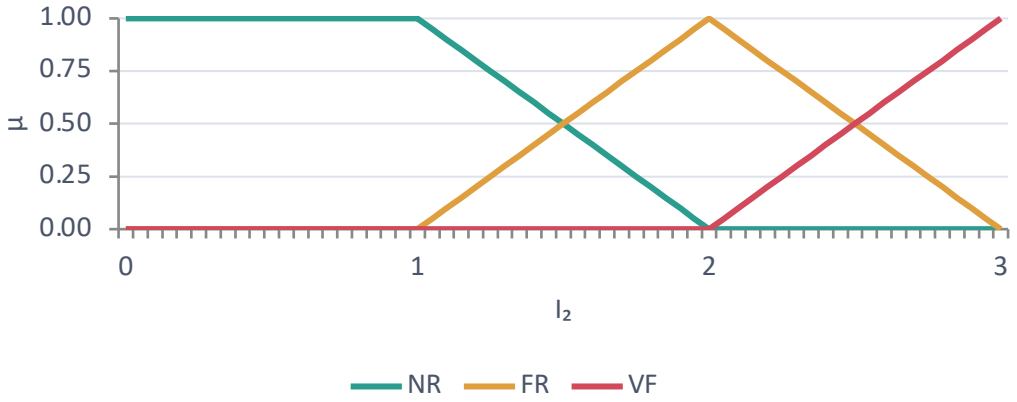
(c) Find the crisp output $y = \Sigma\, w \cdot y^i \,/\, \Sigma\, w$.

Takagi–Sugeno, worked: $I_1 = 6.0$, $I_2 = 2.2$

I_1: L, M, H on [0, 15]



I_2: NR, FR, VF on [0, 3]



$$y^i = a\,I_1 + b\,I_2$$ a = 1, 2, 3 for L, M, H; b = 1, 2, 3 for NR, FR, VF

$I_1 = 6.0$: μL = 0.8, μM = 0.2 **$I_2 = 2.2$: μFR = 0.8, μVF = 0.2**

Rule	IF	$w = \mu \times \mu$	y^i	$w \cdot y^i$
R1	I_1 L, I_2 FR	0.8 × 0.8 = 0.64	10.4	6.656
R2	I_1 L, I_2 VF	0.8 × 0.2 = 0.16	12.6	2.016
R3	I_1 M, I_2 FR	0.2 × 0.8 = 0.16	16.4	2.624
R4	I_1 M, I_2 VF	0.2 × 0.2 = 0.04	18.6	0.744
Total		**Σw = 1**		**Σwy = 12.04**

$y = \Sigma w^i y^i / \Sigma w^i = 12.04 / 1 = 12.04$ (already crisp)

Summary

Relation	$\mu R(x, y) \in [0, 1]$ on $X \times Y$	**Composition**	max over y of $\min(\mu R(x, y), \mu S(y, z))$
Proposition	$T(v \text{ is } F) = \mu F(v)$	**Implication**	Mamdani $a \wedge b$ · Zadeh $(1 - a) \vee (a \wedge b)$
GMP	$B' = A' \circ R$	**GMT**	$A' = B' \circ R$
λ-cut	$\{x \mid \mu(x) \geq \lambda\}$	**CoG**	$\int x\mu \, dx \, / \int \mu \, dx$
Mamdani FLC	fuzzify → α = min → clip → max → defuzzify	**Takagi–Sugeno**	$y = \Sigma w^i y^i \, / \, \Sigma w^i$

- Every step reuses min, max, the complement and the Cartesian product from lecture 4.
- A rule is a relation, inference is a composition, and a controller is inference plus defuzzification.
- Mamdani controllers are easy to read; Takagi–Sugeno controllers are faster and more accurate but need coefficients.

09

Questions

Review questions and practice problems

Review questions

1. Why must R and S share the middle universe Y before $R \circ S$ can be formed?
2. Contrast “A coupled with B” and “A entails B”. What does each give when $\mu A(x) = 0$?
3. State GMP and GMT and the composition used in each.
4. What does a λ-cut return, and why is it not yet a single crisp value?
5. Why can centre of sums differ from centre of gravity on the same rule outputs?
6. Mamdani or Takagi–Sugeno: which needs a defuzzifier, which is easier to read, and why?

Practice problems

7 $R = [0.3\ 0.8; 0.6\ 0.1]$, $S = [0.5\ 0.9; 0.4\ 0.7]$. Find $R \circ S$ by max–min and by max–product.

8 $A = \{(x_1, 0.3), (x_2, 1), (x_3, 0.6)\}$, $B = \{(y_1, 0.7), (y_2, 0.2)\}$. Build $A \times B$ and Zadeh's max–min rule matrix.

9 With R from problem 8 and $A' = \{(x_1, 0.4), (x_2, 0.8), (x_3, 0.9)\}$, find B' by GMP.

10 $T(P) = 0.7$ and $T(Q) = 0.4$. Find $T(\neg P)$, $T(P \wedge Q)$, $T(P \vee Q)$, $T(P \Rightarrow Q)$ and $T(P = Q)$.

11 Young = {(10, 0.5), (20, 0.8), (30, 0.8), (40, 0.5), (50, 0.3)}. Find FoM, LoM, MoM and the discrete CoG.

12 For $R = [0.7\ 0.2\ 0.3; 0.9\ 0.5\ 1; 0.8\ 0.3\ 0.7]$ find the λ-cuts $R_{0.8}$ and $R_{0.3}$.

13 Robot: $D = 0.9$ m and $\theta = -20°$. Fuzzify both inputs, list the fired rules and their strengths α.

14 Takagi–Sugeno example with $I_1 = 8$ and $I_2 = 1.5$: find every w^i and y^i, then y.

References

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- J. R. Raol and S. M. George, Intelligent Estimation: A Soft Computing Paradigm, §5.3.3 fuzzy implication methods, §5.3.4 composition, §5.3.5 defuzzification, §5.4 GMP and GMT.
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- T. Takagi and M. Sugeno, “Fuzzy identification of systems and its applications to modeling and control,” IEEE Trans. Systems, Man, and Cybernetics, vol. 15, no. 1, pp. 116–132, 1985.